INDUCTION MOTOR MODELLING FOR VECTOR CONTROL PURPOSES

Mircea Popescu

Abstract

Widely used in many industrial applications, the induction motors represent the starting point when an electrical drive system has to be designed. In modern control theory, the induction motor is described by different mathematical models, according to the employed control method. In the symmetrical three-phase version or in the unsymmetrical two-phase version, this electrical motor type can be associated with vector control strategy. Through this control method, the induction motor operation can be analysed in a similar way to a DC motor. The goal of this research is to summarize the existing models and to develop new models, in order to obtain a unified approach on modelling of the induction machines for vector control purposes. Starting from vector control principles, the work suggests the *d-q* axes unified approach for all types of the induction motors. However, the space vector analysis is presented as a strong tool in modelling of the symmetrical induction machines. When an electrical motor is viewed as a mathematical system, with inputs and outputs, it can be analysed and described in multiple ways, considering different reference frames and state-space variables. All the mathematical possible models are illustrated in this report. The suggestions for what model is suitable for what application, are defined as well. As the practical implementation of the vector control strategies require digital signal processors (DSP), from the continuos time domain models are derived the discrete time domain models. The discrete models permit the implementation of the mathematical model of the induction motors, in order to obtain high efficiency sensorless drives. The stability of these various models is analysed.

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List of principal symbols:

Scalar variables are denoted by plane letters. Vector variables are denoted by underlined letters. Boldface symbols are used for matrix variables.

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$	state-space coefficient matrix
a_{ij}	coefficients for state-space variables
B_{m}	viscous friction coefficient
H	relative inertia constant
$\boldsymbol{H}$	transfer function matrix
$I_{(a,b,c)s}$	instantaneous stator phase current for the three-phase induction machine
$I_{(a,b,c)r}$	instantaneous rotor phase current for the three-phase induction machine
$\tilde{I}_{(d,q)s}$	complex stator phase currents in d-q co-ordinates for steady-state analysis
$\tilde{I}_{(a,b)s}$	complex stator phase currents in physical co-ordinates for steady-state analysis
$\tilde{I}_{(d,q)r}$	complex rotor phase currents in d-q co-ordinates for steady-state analysis
$\tilde{I}_{(a,b)r}$	complex rotor phase currents in physical co-ordinates for steady-state analysis
$\underline{I}_{\mathrm{s}}$	stator current space vector
$\underline{I}_{\mathrm{r}}$	rotor current space vector
$i_{(a,b,c)s}$	instantaneous stator phase currents for the three-phase induction machine
$i_{(a,b,c)r}$	instantaneous rotor phase currents for the three-phase induction machine
$i_{(a,b)s}$	instantaneous stator phase currents for the single-phase induction machine
$i_{(a,b)r}$	instantaneous rotor phase currents for the single-phase induction machine
$i_{(A,B,C)s}$	phase currents for a three-phase system
$i_{(X,Y)s}$	phase currents for an orthogonal two-phase system
$i_{(d,q)s}$	instantaneous stator phase currents in d-q co-ordinates
$i'_{(d,q)r}$	instantaneous referred rotor phase currents in d-q co-ordinates
$i'_{(d,q)R}$	Γ models instantaneous referred rotor phase currents in d-q co-ordinates
$i_{\mathrm{s}(R,I)}$	stator phase currents in complex co-ordinates and per unit system
$i'_{\mathrm{r}(R,I)}$	rotor phase currents in complex co-ordinates and per unit system
$\underline{i}_{\mathrm{s}}$	stator current space vector in per unit system
$\underline{i}'_{\mathrm{r}}$	referred rotor current space vector in per unit system
J	inertia constant
j	complex operator
$K_{\mathrm{p,i}}$	proportional, respectively integrative constant for PI controllers
$\boldsymbol{K}^{\mathrm{s,r}}_{\mathrm{s,r}}$	transformation matrix from abc co-ordinates to d-q co-ordinates
k	turns ratio for the unsymmetrical single-phase machine
$L_{(a,b,c)(s,r)}$	self-inductance for stator phase, respectively rotor

$L_{l(s,r)}$	leakage inductance for symmetrical stator phase, respectively rotor
$L_{l(m,a)}$	leakage inductance for unsymmetrical stator phase: main, respectively auxiliary
l_M	magnetisation inductance in per unit system
l_s	total stator inductance in per unit system
l'_r	total rotor inductance in per unit system
$M_{(a,b,c),(s,r)}$	mutual inductance for stator phase, respectively rotor
$N_{s,r}$	turns number for stator phase, respectively rotor
P	number of poles
p	derivative operator
R_s	per unit stator phase resistance for the three-phase induction machine
r_s	stator phase resistance for the symmetrical induction machine
r_m	main stator phase resistance for the unsymmetrical single-phase machine
r_a	auxiliary stator phase resistance for the unsymmetrical single-phase machine
r_r	rotor phase resistance for the symmetrical induction machine
r'_r	referred rotor phase resistance for the symmetrical induction machine
s	Laplace operator
s_m	critical slip for the induction machine
T	sampling period
T_e	instantaneous electromagnetic torque
T_L	load torque
t	time
$U_{(a,b,c)s}$	instantaneous stator phase voltage for the three-phase induction machine
$U_{(a,b,c)r}$	instantaneous rotor phase voltage for the three-phase induction machine
$U_{s(R,I)}$	stator phase voltage in complex co-ordinates and per unit system
$\widetilde{U}_{(d,q)s}$	complex stator phase voltages in d-q co-ordinates for steady-state analysis
$\widetilde{U}_{(a,b)s}$	complex stator phase voltages in physical co-ordinates for steady-state analysis
$\underline{U}_s$	stator voltage space vector
$\underline{U}_r$	rotor voltage space vector
$u_{(d,q)s}$	instantaneous stator phase voltages in d-q co-ordinates
$u'_{(d,q)r}$	instantaneous rotor phase voltages in d-q co-ordinates
$\underline{u}_s$	stator voltage space vector in per unit system
$\underline{u}'_r$	referred rotor voltage space vector in per unit system
W_c	magnetic coenergy
X_b	base variable value for per unit system
$\underline{x}$	space vector variable
x_m	magnetisation reactance
x_{ls}	stator phase leakage reactance for the symmetrical induction machine

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$x_{l(m,a)}$	stator phase leakage reactance for the unsymmetrical induction machine: main, respectively auxiliary
x'_{lr}	rotor leakage reactance
x_L	Γ models equivalent leakage reactance for symmetrical induction machine
$x_{L(d,q)}$	Γ models equivalent leakage reactance for unsymmetrical induction machine
x_M	Γ models equivalent magnetisation reactance for symmetrical induction machine
$x_{M(d,q)}$	Γ models equivalent magnetisation reactance for unsymmetrical induction machine
z	discrete co-ordinate
α	space vector operator
γ	Γ models turns ratio for symmetrical induction machine
$\gamma_{(d,q)}$	Γ models turns ratio for unsymmetrical induction machine
$\Delta_{R,I}$	real, respectively imaginary component of a matrix determinant
$\psi_{(d,q)s}$	stator flux linkage in d-q co-ordinates and per unit system
$\psi_{(d,q)m}$	magnetising flux linkage in d-q co-ordinates and per unit system
$\psi_{(d,q)r}$	rotor flux linkage in d-q co-ordinates and per unit system
$\psi_{s(R,I)}$	stator flux linkage in complex co-ordinates and per unit system
$\psi'_{r(R,I)}$	rotor flux linkage in complex co-ordinates and per unit system
$\lambda_{(d,q)s}$	stator flux linkage in d-q co-ordinates and flux linkage units per second.
$\lambda_{(a,b,c)s}$	stator flux linkage fluxes in abs or ab co-ordinates
$\lambda'_{(d,q)r}$	referred rotor linkage fluxes in d-q co-ordinates and flux linkage units per second
$\lambda'_{(d,q)R}$	Γ models referred rotor linkage fluxes in d-q co-ordinates and flux linkage units per second
$\lambda_{(a,b,c)r}$	rotor flux linkage in abs or ab co-ordinates
$\underline{\lambda}'_r$	referred rotor flux linkage space vector for three-phase induction machine
$\underline{\lambda}_s$	stator flux linkage space vector for three-phase induction machine
$\underline{\lambda}_\gamma$	arbitrarily flux linkage space vector for three-phase induction machine
$\Re_m$	magnetic reluctance
$\sigma_{s,r}$	leakage factor for stator phase, respectively rotor phase
θ_r	periferical displacement between stator and rotor space vectors
θ_k	periferical displacement between stator and arbitrary space vectors
Ω	relative angular frequency in per unit system
ω	angular frequency of the supply system
ω_b	base angular frequency of the supply system
ω_n	rated angular frequency of the induction machine in electrical degrees
ω_r	rotor angular frequency in electrical degrees

1. VECTOR CONTROL OF INDUCTION MOTORS - OVERVIEW

1.1.Introduction

The electrical DC drive systems are still used in a wide range of industrial applications, although they are less reliable than the AC drives. Their advantage consists in simple and precise command and control structures.

The AC drives, sometimes more expensive but far more reliable, (Rajashekara et al. 1996) require complex modern control techniques. The design of a control system is realised in two important steps:

1. The drive system has to be converted into a mathematical model, in order to accomplish the analysis and the evaluation of the system.
2. The imposed response of the drive system is obtained through an optimal regulator, when external perturbations are present.

The induction motors are relatively cheap and rugged machines because their construction is realised without slip rings or commutators. These advantages have determined an important development of the electrical drives, with induction machine as the execution element, for all related aspects: starting, braking, speed reversal, speed change, etc. The dynamic operation of the induction machine drive system has an important role on the overall performance of the system of which it is a part.

There are two fundamental directions for the induction motor control:

- Analogue: direct measurement of the machine parameters (mainly the rotor speed), which are compared to the reference signals through closed control loops;
- Digital: estimation of the machine parameters in the sensorless control schemes (without measuring the rotor speed), with the following implementation methodologies:
  - Slip frequency calculation method;
  - Speed estimation using state equation;
  - Estimation based on slot space harmonic voltages;
  - Flux estimation and flux vector control;
  - Direct control of torque and flux;
  - Observer-based speed sensorless control;
  - Model reference adaptive systems;
  - Kalman filtering techniques;
  - Sensorless control with parameter adaptation;
  - Neural network based sensorless control;
  - Fuzzy-logic based sensorless control.

Another classification of the control techniques for the induction machine is made by Holtz (1998) from the point of view of the controlled signal:

a) Scalar control:

a.1 Voltage/frequency (or v/f) control;

a.2 Stator current control and slip frequency control. These techniques are mainly implemented through direct measurement of the machine parameters.

b) Vector control:

b.1 Field orientation control (FOC): b.1.1. Indirect method; b.1.2. Direct method;

b.2 Direct torque and stator flux vector control. These techniques are realised both in analogue version (direct measurements) and digital version (estimation techniques)

The development of accurate system models is fundamental to each stage in the design, analysis and control of all electrical machines. The level of precision required of these models depends entirely on the design stage under consideration. In particular, the mathematical description used in

machine design requires very fine tolerance levels as stated by Nabae et al. (1980) and Murata et al (1990). However, in the development of suitable models for control purposes, it is possible to make certain assumptions that considerably simplify the resulting machine model. Nonetheless, these models must incorporate the essential elements of both the electromagnetic and the mechanical systems for both steady state and transient operating conditions (Nowotny and Lipo - 1996). Additionally, since modern electric machines are invariably fed from switching power conversion stages, the developed motor models should be valid for arbitrary applied voltage and current waveforms. This work presents suitable models for use in digital current control of the induction motors. In addition, the limits of the validity of these models are summarised and, in some cases, the models are extended to account for some non-idealities of the machine.

Usually, the following assumptions are made (Lorenz et al. 1994):

- No magnetic saturation, i.e. machine inductance is not affected by current level.
- No saliency effects i.e. machine inductance are not functions of position.
- Negligible spatial mmf harmonics i.e. stator windings are arranged to produce sinusoidal mmf distributions.
- The effects of the stator slots may be neglected.
- There is no fringing of the magnetic circuit.
- The magnetic field intensity is constant and radially directed across the air-gap.
- Eddy current and hysteresis effects are negligible.

The modern control theory for an electrical drive system requires the existence of a real-time, stabile, and precise mathematical model for each component of the system. The analysis and the design of the numerical command for such systems depend on the hardware and software resources. If in communication techniques the real-time response of the system is not always compulsory, in industrial processes the real-time response of the drive systems is essential.

The soft numerical command for the electrical drive systems is far more flexible to implement than the hardware version. For the latter, lately there is an intense research effort for implementing ASIC (application specific integrated circuit). The numerical command of the electrical drive systems is a challenging task mainly due to the DSP (digital signal processing) technology. Now it is possible to realise linear and non-linear techniques for implementing continuos and discrete mathematical models of the entire element of an electrical drive system, including the electrical machine (Xu and Nowotny - 1990, 1992).

For the AC drives there are several solutions for implementing the command and the control of the system. A quick summary of the existing technologies already out there in the field is given below:

DC Drives

Initially the DC drives were used for variable speed control because they could easily achieve a good torque and speed response with high accuracy. Field orientation of the motor is achieved using a mechanical commutator with brushes. In DC, torque is controlled using the armature current and field current. The main drawback of this technique is the reduced reliability of the DC motor - the fact that brushes and commutators wear down and need regular servicing; that DC motors can be costly to purchase; and that they require encoders for positional feedback.

AC Drives

The evolution of AC variable speed drive technology has been partly driven by the desire to emulate the performance of the DC drive, such as fast torque response and speed accuracy, while using robust, cheap to purchase and relatively maintenance-free AC motors (Kelemen and Imecs - 1987).

AC Drives, frequency controlled using PWM

With this technique, sometimes known as scalar control, the field orientation of the motor is not used. Instead, the frequency and the voltage are the main control variables and are applied to the stator windings. The status of the rotor is ignored, meaning that no speed or position signal is fed

back. The drive is therefore regarded as an open-loop drive. This type of drive is suitable for applications such as pumps and fans, which do not require high levels of accuracy or precision.

AC Drives, flux vector control using PWM

Here, field orientation is achieved by mathematical modelling using microprocessors and feedback of rotor speed and angular position relative to the stator field by means of an encoder (Vas - 1990). This results in a drive with greater stability and capable of fast torque response and accurate speed control. But the drawback is the need for the encoder, which reduces drive system reliability and adds cost. The controlling variables in a DC drive for torque are armature current and field current, and armature voltage for torque. AC drives using the PWM principle; however, use voltage and frequency as the controlling variables and these are controlled by a device called a modulator. A modulator adds considerable delay in the responsiveness of a motor to changes in torque and speed. Furthermore, with flux vector AC drives, a tacho-generator or position encoder is invariably needed to obtain any real degree of accuracy. Such devices are costly and compromise the simplicity of the AC induction motor.

AC Drives, sensorless flux vector

The flux vector controlled drive with encoder feedback does offer very high levels of performance across a wide power range and should not be confused with sensorless vector - or open loop vector - drives, which offer performance only slightly superior to that of a standard inverter using scalar control (Rajashekara et al. - 1996).

1.2.Algorithm of vector control

The induction motors are very common because they are inexpensive and robust, finding use in everything from industrial applications such as pumps, fans, and blowers to home appliances. Traditionally, induction motors have been run at a single speed, which was determined by the frequency of the main voltage and the number of poles in the motor. Controlling the speed of an induction motor is far more difficult than controlling the speed of a DC motor since there is no linear relationship between the motor current and the resulting torque as there is for a DC motor.

The technique called vector control can be used to vary the speed of an induction motor over a wide range. It was initially developed by Blaschke (1971-1973). In the vector control scheme, a complex current is synthesised from two quadrature components, one of which is responsible for the flux level in the motor, and another which controls the torque production in the motor. Essentially, the control problem is reformulated to resemble the control of a DC motor. Vector control offers a number of benefits including speed control over a wide range, precise speed regulation, fast dynamic response, and operation above base speed.

The vector control algorithm is based on two fundamental ideas. The first is the flux and torque producing currents. An induction motor can be modelled most simply (and controlled most simply) using two quadrature currents rather than the familiar three phase currents actually applied to the motor. These two currents called direct (I_d) and quadrature (Iq) are responsible for producing flux and torque respectively in the motor. By definition, the I_q current is in phase with the stator flux, and I_d is at right angles. Of course, the actual voltages applied to the motor and the resulting currents are in the familiar three-phase system. The move between a stationary reference frame and a reference frame, which is rotating synchronous with the stator flux, becomes then the problem. This leads to the second fundamental idea behind vector control.

The second fundamental idea is that of reference frames. The idea of a reference frame is to transform a quantity that is sinusoidal in one reference frame, to a constant value in a reference frame, which is rotating at the same frequency. Once a sinusoidal quantity is transformed to a constant value by careful choice of reference frame, it becomes possible to control that quantity with traditional proportional integral (PI) controllers.

Vector transforms

The Park and Clarke vector transforms are one of the keys to vector control of induction motors.

I) Clarke transform

The forward Clarke (1943) transform does a magnitude invariant translation from a three phase system into two orthogonal components. If the neutral - ground connection is neglected, the variables in a three-phase system (A, B, and C) sum is equal to zero, and there is a redundant information. Therefore, the system can be reduced to two variables, called X and Y. The Clarke transform is given by:

$$\begin{bmatrix} i_{Xs}(t) \\ i_{Ys}(t) \end{bmatrix} = \frac{2}{3} \cdot \begin{bmatrix} 1 & \cos(\gamma) & \cos(2\gamma) \\ 0 & \sin(\gamma) & \sin(2\gamma) \end{bmatrix} \cdot \begin{bmatrix} i_{As}(t) \\ i_{Bs}(t) \\ i_{Cs}(t) \end{bmatrix} \tag{1}$$

where:

$$\gamma = \frac{2\pi}{3}$$

Using the relation:

$$i_{As}(t) + i_{Bs}(t) + i_{Cs}(t) = 0 \tag{2}$$

and the fact that:

$$\cos\left(\frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2} \tag{3}$$

Thus, the Clarke transform can be simplified to:

$$\begin{bmatrix} i_{Xs}(t) \\ i_{Bs}(t) \end{bmatrix} = \begin{bmatrix} i_{As}(t) \\ \frac{1}{\sqrt{3}} \cdot (i_{As}(t) - i_{Cs}(t)) \end{bmatrix} \tag{4}$$

The Clarke transform can also be understood using a vector diagram as shown in Fig. 1.1. In the figure, A, B, and C are the axes of a three phase system, each offset 120° from the other. X and Y are the axes of a two variable system where X is chosen to be coincident with A. To perform the Clarke transform of a three variable system (i_A, i_B, i_C), i_X is equal to i_A and i_Y is the scaled projection of i_B and i_C onto the Y axis. The scaling is necessary to preserve the signal magnitudes through the transform.
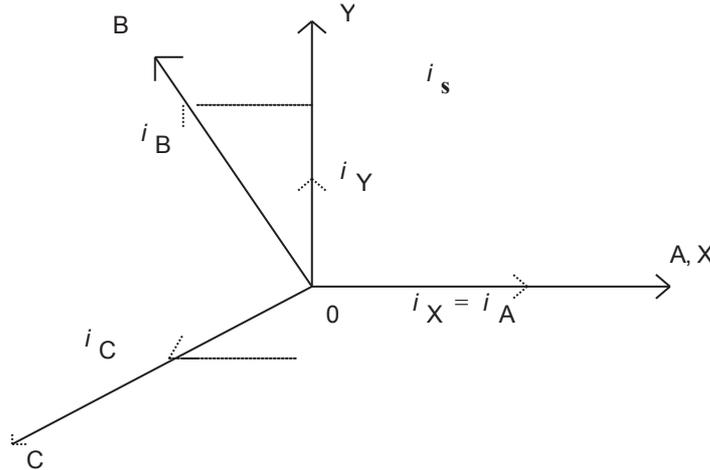
Fig. 1.1. Clarke Transform Vector Diagram

The Clarke transform preserves the magnitude, and realise a quadrature between the current components.

II) Park transform

The Park (1929) transform is a vector rotation, which rotates a vector (defined by its quadrature components) through a specified angle. The Park transform function implements the following set of equations:

$$\begin{bmatrix} Out_x(t) \\ Out_y(t) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} In_x(t) \\ In_y(t) \end{bmatrix} \tag{5}$$

where θ is the angle to rotate the vector through. A reverse vector rotation can be accomplished simply by changing the sign on the sin (θ) input value. The vector rotation is illustrated by Fig. 1.2.

Some references (Vas -1990, Nowotny and Lipo - 1996) describe the Park transform as a combination of the Clark and Park transforms presented here. Breaking into a three-variable-to-two transform (i.e. the Clarke transform) and a vector rotation is done for efficiency of calculation: with separate Park and Clarke transforms, only two trigonometric calculations are required as opposed to 6 in the traditional Park transform.
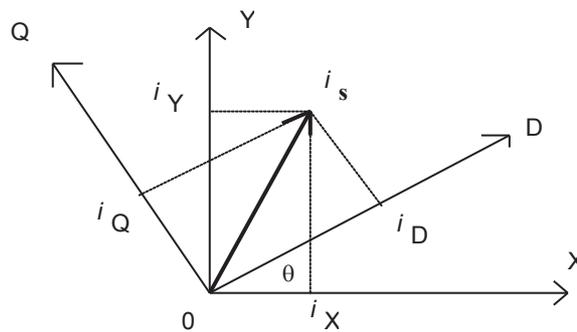
Fig.1.2. Park transform vector diagram

1. 3. Field orientation control (FOC)

Vector control techniques have made possible the application of induction motors for high-performance applications where traditionally only DC drives were applied (Holtz - 1995). The vector control scheme enables the control of the induction motor in the same way as separately excitation DC motors. As in the DC motor, torque control of induction motor is achieved by controlling the torque current component and flux current component independently. The basic schemes of indirect and direct methods of vector control are shown in Figs. 1.3 –1.5. The direct vector control method depends on the generation of unit vector signals from the stator or air-gap flux signals. The air-gap signals can be measured directly or estimated from the stator voltage and current signals. The stator flux components can be directly computed from stator quantities. In these systems, rotor speed is not required for obtaining rotor field angle information. In the indirect vector control method, the rotor field angle and thus the unit vectors are indirectly obtained by summation of the rotor speed and slip frequency.
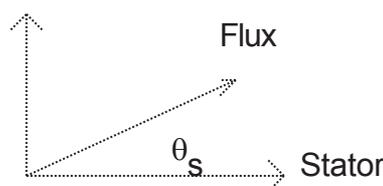
Fig. 1.3. Position of the rotor flux vector
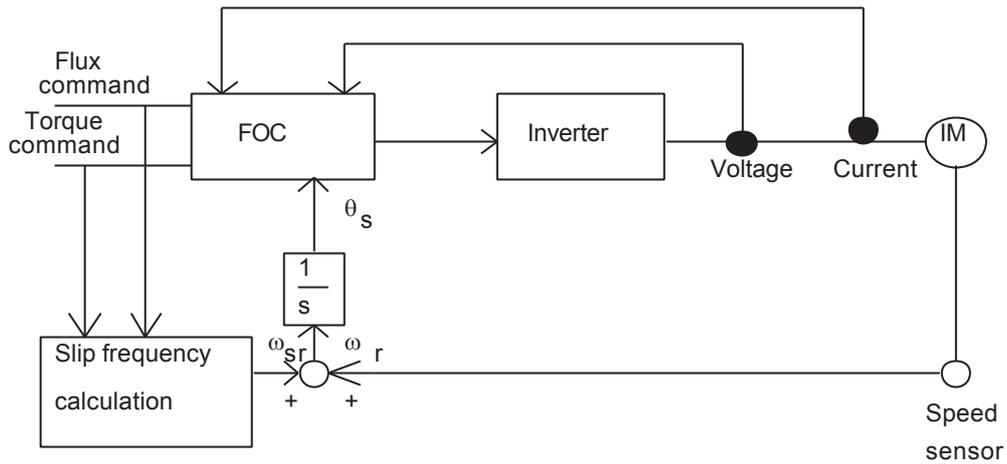
Fig. 1.4. Indirect vector control method
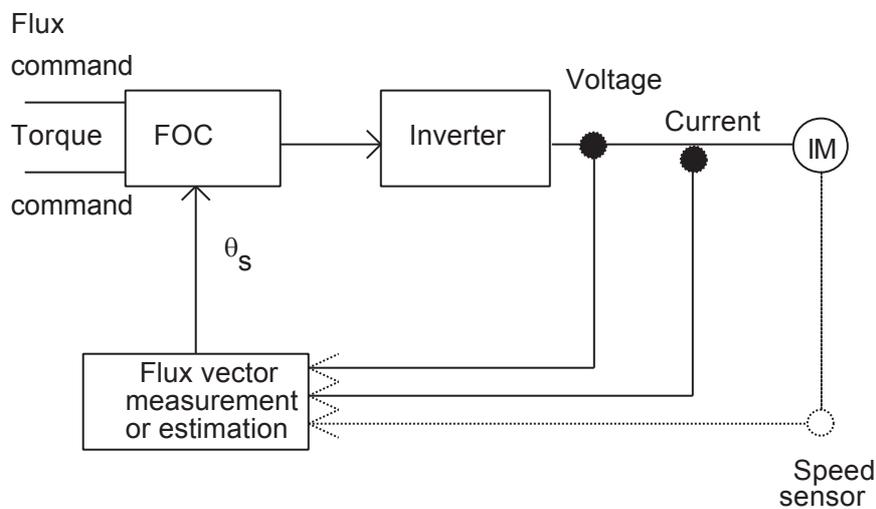
Fig. 1.5. Direct vector control method

Fundamental requirements for the FOC are the knowledge of two currents (if the induction motor is star connected) and the rotor flux position. Knowledge of the rotor flux position is the core of the FOC. In fact if there is an error in this variable the rotor flux is not aligned with d-axis and the current components are incorrectly estimated. In the induction machine the rotor speed is not equal to the rotor flux speed (there is a slip speed; as such, a special method to calculate the rotor flux position (angle) is needed. The basic method is the use of the current model.

Thanks to FOC it becomes possible to control, directly and separately, the torque and flux of the induction motors. Field oriented controlled induction machines obtain every DC machine advantage: instantaneous control of the separate quantities allowing accurate transient and steady-state management.

1. 4. Direct torque control

The most modern technique is direct torque and stator flux vector control method (DTC). It has been realised in an industrial way by ABB, by using the theoretical background proposed by Blashke and Depenbrock during 1971-1985. This solution is based both on field oriented control (FOC) as well as on the direct self-control theory.

Starting with a few basics in a variable speed drive the basic function is to control the flow of energy from the mains to a process via the shaft of a motor. Two physical quantities describe the state of the shaft: torque and speed. Controlling the flow of energy depends on controlling these

quantities. In practice either one of them is controlled and we speak of "torque control" or "speed control". When a variable speed drive operates in torque control mode the speed is determined by the load. Torque is a function of the actual current and actual flux in the machine. Likewise when operated in speed control the torque is determined by the load.

Variable speed drives are used in all industries to control precisely the speed of electric motors driving loads ranging from pumps and fans to complex drives on paper machines rolling mills cranes and similar drives.

The idea is that motor flux and torque are used as primary control variables which is contrary to the way in which traditional AC drives control input frequency and voltage, but is in principle similar to what is done with a DC drive, where it is much more straightforward to achieve. In contrast, traditional PWM and flux vector drives use output voltage and output frequency as the primary control variables but these need to be pulse width modulated before being applied to the motor. This modulator stage adds to the signal processing time and therefore limits the level of torque and speed response time possible from the PWM drive.

In contrast, by controlling motor torque directly, DTC provides dynamic speed accuracy equivalent to closed loop AC and DC systems and torque response times that are 10 times faster. It is also claimed that the DTC does not generate noise like that produced by conventional PWM AC drives. And the wider spectrum of noise means that amplitudes are lower which helps to control EMI and RFI emissions. The basic structure of direct torque and stator flux vector control is presented in Fig. 1.6.
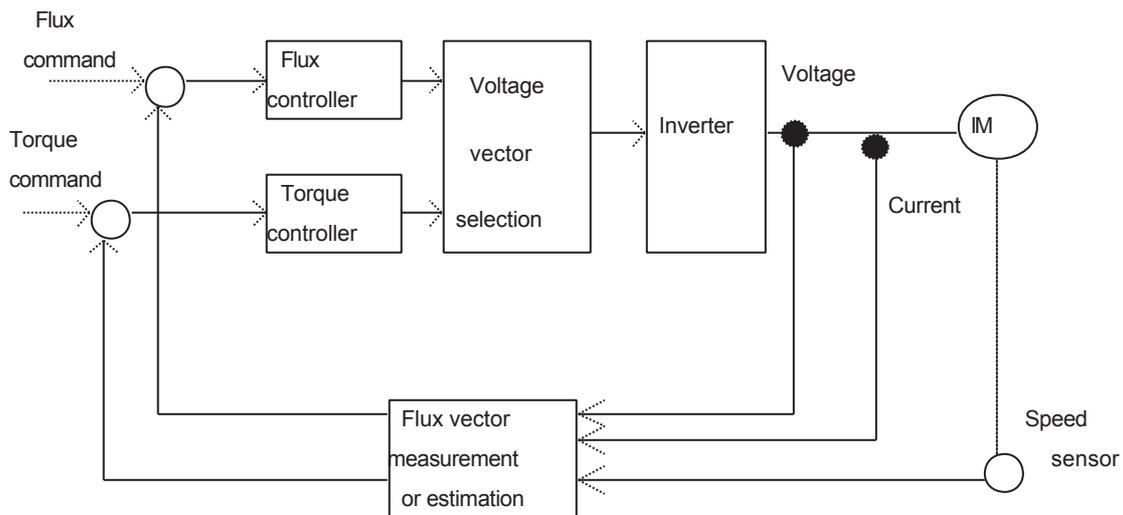
Fig. 1.6. Basic structure of direct torque and flux vector control

In DTC field orientation is achieved without feedback using advanced motor theory to calculate the motor torque directly and stator flux without using a modulator or a requirement for a tachogenerator or position encoder to feed back the speed or position of the motor shaft. Both parameters are obtained instead from the motor itself. DTC's configuration also relies on two key developments - the latest high-speed signal processing technology and a highly advanced motor model precisely simulating the actual motor within the controller. A DSP (digital signal processor) is used together with ASIC hardware to determine the switching logic of the inverter.

The motor model is programmed with information about the motor, which enables it to determine parameters including stator resistance, mutual inductance saturation coefficients and motor inertia. The model also encompasses temperature compensation, which is essential for good static speed accuracy without encoder.

In normal operation, measurements of the two motor phase currents and the drive DC link voltage, together with information about the switching state of the inverter are fed into the motor model . The motor model then outputs control signals, which are accurate estimates of the actual motor torque and actual stator flux. All control signals are transmitted via optical links for high speed. In this way, the semiconductor switching devices of the inverter are supplied with an optimum switching pattern for reaching or maintaining an accurate motor torque.

Also, both shaft speed and electrical frequency are calculated within the motor model. There is no need to feedback any shaft speed or position with tachometers or encoders to meet the demands of 95% of industrial applications. However, there will always be some special applications where even greater speed accuracy will be needed and when the use of an encoder improves the accuracy of speed control in DTC. But even then, the encoder does not need to be as costly or as accurate as the one used in traditional flux vector drives, as DTC only has to know the error in speed, not the rotor position.

The drive will have a torque response time typically better than 5ms. This compares with response for both flux vector PWM drives and DC drives fitted with encoders. The newer sensorless flux vector drives now being launched by other drives manufacturers have a torque response measured in hundreds of milliseconds.

DTC also provides exceptional torque control linearity. For the first time with an open loop AC drive, torque control can be obtained at low frequencies, including zero speed, where the nominal torque step can be increased in less than 1ms. The dynamic speed accuracy of DTC drives is better than any open loop AC drives and comparable to DC drives, which use feedback.

DTC brings other special functions, not previously available with AC drives, including automatic starting in all motor electromagnetic and mechanical states. There is no need for additional parameter adjustments, such as torque boost or starting mode selection, such as flying start. DTC control automatically adapts itself to the required condition. In addition, based on exact and rapid control of the drive intermediate DC link voltage, DTC can withstand sudden load transients caused by the process, without any overvoltage or overcurrent trip.

2. CONTINUOUS-TIME DOMAIN LINEAR MODELS OF THE THREE-PHASE INDUCTION MACHINE

2.1. Introduction

Until the last decades the three-phase induction machine was mainly used in constant speed drives due to the control system performance, not to the operating principle of the machine.

Nowadays, this situation is completely changed. With the technical progress in power electronics and microelectronics, the three-phase induction machine control becomes very flexible and highly efficient. Since 1983, the year when the digital signal processor (DSP) appeared, the control theory for this type of machine was permanently improved.

New mathematical models have to be implemented for the three-phase induction machine in order to analyse its operation both dynamically and in steady-state.

2.2. Voltage and flux linkage equations

The first mathematical model for the dynamic analysis of the induction machine was based on the two real axis reference frame, developed initially by Park (1929) for the synchronous machine. Using the symmetric configuration of the induction machine, Kovacs and Racz (1959) have elaborated the space complex vector theory, and obtained a model for the steady-state analysis of the machine. Both theories are used for modelling the three-phase induction machine. The following assumptions are made when a complete equations system is written to describe the continuous-time linear model of the induction machine (Krause et al. 1995):

- Geometrical and electrical machine configuration is symmetrical;
- Space harmonics of the stator and rotor magnetic flux are negligible;
- Infinitely permeable iron;
- Stator and rotor windings are sinusoidally distributed in space and replaced by an equivalent concentrated winding;
- Saliency effects, the slotting effects are neglected;
- Magnetic saturation, anisotropy effect, core loss and skin effect are negligible;
- Windings resistance and reactance do not vary with the temperature;
- Currents and voltages are sinusoidal terms.
- End and fringing effects are neglected

All these assumptions do not alter in a serious way the final result for a wide range of induction machines.
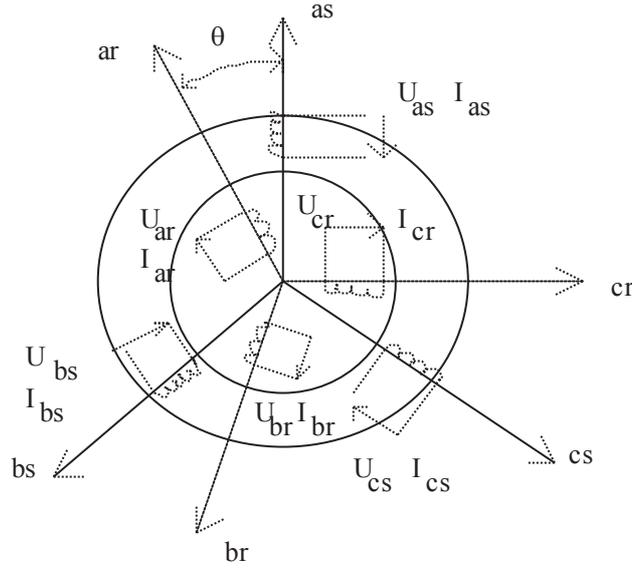
Fig. 2.1. The real model of the three-phase induction machine
with three stator windings and three rotor windings

For the machine stator in Fig.2.1 if we choose the stator reference frame, the voltage equations are as follows:

$$U_{as} = r_s I_{as} + \frac{d\lambda_{as}}{dt}$$

$$U_{bs} = r_s I_{bs} + \frac{d\lambda_{bs}}{dt} \qquad (6\text{-}8)$$

$$U_{cs} = r_s I_{cs} + \frac{d\lambda_{cs}}{dt}$$

where U_{as}, U_{bs}, U_{cs} are the instantaneous stator voltages, I_{as}, I_{bs}, I_{cs} are the instantaneous stator currents, $r_s = r_{as} = r_{bs} = r_{cs}$ is the stator winding resistance and λ_{as}, λ_{bs}, λ_{cs} are the total magnetic fluxes for the three stator windings.

The flux-current relations are determined after detailing the total flux of a stator winding. For the other two windings, there are valid similar relations:

$$\lambda_{as} = \lambda_{asas} + \lambda_{bsas} + \lambda_{csas} + \lambda_{aras} + \lambda_{bras} + \lambda_{cras} \qquad (9)$$

where the flux components are:

λ_{asas} the magnetic flux produced by stator phase current *as* in the stator phase winding *as*

λ_{bsas} the magnetic flux produced by stator phase current *bs* in the stator phase winding *as*

λ_{csas} the magnetic flux produced by stator phase current *cs* in the stator phase winding *as*

λ_{aras} the magnetic flux produced by rotor phase current *ar* in the stator phase winding *as*

λ_{bras} the magnetic flux produced by rotor phase current *br* in the stator phase winding *as*

λ_{cras} the magnetic flux produced by rotor phase current *cr* in the stator phase winding *as*

These components are computed with the expressions:

$$\lambda_{asas} = L_{as} I_{as} \qquad\qquad \lambda_{aras} = M_{aras} I_{ar}$$

$$\lambda_{bsas} = M_{bsas} I_{bs} \qquad\qquad \lambda_{bras} = M_{bras} I_{br}$$

$$\lambda_{csas} = M_{csas} I_{cs} \qquad\qquad \lambda_{cras} = M_{cras} I_{cr}$$

Self-inductance L_{as} has two components, one created by the linkage magnetic flux L_{mas} and the second created by the leakage magnetic flux L_{las}:

$$L_{as} = L_{mas} + L_{las} \tag{10}$$

The mutual inductance, which is considered to be equal due to the machine symmetry, can also be split in two components. However, the leakage flux created component in the mutual inductance can be neglected. It results that:

$$M_{asbs} = M_{bsas} = M_{ascs} = M_{csas} = -\frac{1}{2} L_{mas} \tag{11}$$

The mutual inductance within the stator and rotor windings varies with the relative space position between them. The stator flux created by current from rotor phase *ar* in stator phase *as* depend on the angle value θ:

$$M_{aras} = M_{asar} = \frac{w_2 k_{w2}}{w_s k_{ws}} \cdot L_{mas} \cdot \cos\theta = \frac{1}{k_t} \cdot L_{mas} \cdot \cos\theta \tag{12}$$

where k_t represents the turn ratio multiplied by the winding factor ratio. In a similar way the relations for the others mutual inductances can be written:

$$M_{bras} = M_{asbr} = \frac{1}{k_t} \cdot L_{mas} \cdot \cos\left(\theta + \frac{2\pi}{3}\right)$$

$$M_{cras} = M_{ascr} = \frac{1}{k_t} \cdot L_{mas} \cdot \cos\left(\theta + \frac{4\pi}{3}\right)$$

$$\tag{13-14}$$

Due to the symmetrical configuration of the induction machine, we can deduce the total magnetic flux for the stator phase winding *as* expressed as follows:

$$\lambda_{as} = \left(L_{mas} + L_{las}\right) \cdot I_{as} - \frac{1}{2} \cdot L_{mas} I_{bs} - \frac{1}{2} \cdot L_{mas} I_{cs} + \frac{1}{k_t} \cdot L_{mas} I_{ar} \cos\theta +$$

$$\frac{1}{k_t} \cdot L_{mas} I_{br} \cos\left(\theta + \frac{2\pi}{3}\right) + \frac{1}{k_t} \cdot L_{mas} I_{cr} \cos\left(\theta + \frac{4\pi}{3}\right) \tag{15}$$

For the rotor windings, by using a rotor reference frame, it can be developed a similar equation system to the stator case:

$$U_{ar} = r_r I_{ar} + \frac{d\lambda_{ar}}{dt}$$

$$U_{br} = r_r I_{br} + \frac{d\lambda_{bar}}{dt} \tag{16-18}$$

$$U_{cr} = r_r I_{cr} + \frac{d\lambda_{cr}}{dt}$$

where: U_{ar}, U_{br}, U_{cr} are the instantaneous rotor voltages, I_{ar}, I_{br}, I_{cr} are the instantaneous rotor currents, $r_r = r_{ar} = r_{br} = r_{cr}$ is the rotor winding resistance and $\lambda_{ar}, \lambda_{br}, \lambda_{cr}$ are the total magnetic fluxes for the three rotor windings.

The total rotor magnetic flux for the winding *ar* is described by:

$$\lambda_{ar} = \lambda_{arar} + \lambda_{brar} + \lambda_{crar} + \lambda_{asar} + \lambda_{bsar} + \lambda_{csar} =$$

$$= L_{ar} I_{ar} + M_{brar} I_{br} + M_{crar} I_{cr} + M_{asar} I_{as} + M_{bsar} I_{bs} + M_{csar} I_{cs} \tag{19}$$

In this case the mutual inductance is:

$$M_{\text{brar}} = M_{\text{arbr}} = -\frac{1}{2}L_{\text{mar}}$$

$$M_{\text{crar}} = M_{\text{arcr}} = -\frac{1}{2}L_{\text{mar}}$$

$$\text{(20-21)}$$

$$M_{\text{asar}} = M_{\text{aras}} = k_{\text{t}} \cdot L_{\text{mar}} \cos\theta$$

$$M_{\text{bsar}} = M_{\text{arbs}} = k_{\text{t}} \cdot L_{\text{mar}} \cos\left(\theta + \frac{2\pi}{3}\right)$$

$$\text{(22-24)}$$

$$M_{\text{csar}} = M_{\text{arcs}} = k_{\text{t}} \cdot L_{\text{mar}} \cos\left(\theta + \frac{4\pi}{3}\right)$$

Due to the symmetrical windings and motor configuration, one can write the following relation:

$$\frac{1}{k_{\text{t}}} \cdot L_{\text{mas}} = k_{\text{t}} \cdot L_{\text{mar}} = L_{\text{m}} \tag{25}$$

Through a similar algorithm as that one for the stator and rotor phase *as*, respectively *ar*, it is possible to obtain another four equations: two for the stator phases *bs*, and *cs*, and two for the rotor phases *br* and *cr*. All six final equations can be grouped in a matrix form as follows:

$$[U_{\text{s}}] = r_{\text{s}}[I_{\text{s}}] + \frac{d[\lambda_{\text{s}}]}{dt}$$

$$[U_{\text{r}}] = r_{\text{r}}[I_{\text{r}}] + \frac{d[\lambda_{\text{r}}]}{dt}$$

$$\text{(26-29)}$$

$$[\lambda_{\text{s}}] = [L_{\text{s}}] \cdot [I_{\text{s}}] + [M_{\text{s}}] \cdot [I_{\text{r}}]$$

$$[\lambda_{\text{r}}] = [L_{\text{r}}] \cdot [I_{\text{r}}] + [M_{\text{r}}] \cdot [I_{\text{s}}]$$

where: $[U_{\text{s}}], [U_{\text{r}}], [I_{\text{s}}], [I_{\text{r}}], [\lambda_{\text{s}}], [\lambda_{\text{r}}]$ represent the transpose matrix for the stator and rotor voltage, current, respectively flux vectors. As an example are given the flux matrix:

$$[\lambda_{\text{s}}] = [\lambda_{\text{as}} \quad \lambda_{\text{bs}} \quad \lambda_{\text{cs}}]^{\text{T}} \quad [\lambda_{\text{r}}] = [\lambda_{\text{ar}} \quad \lambda_{\text{br}} \quad \lambda_{\text{cr}}]^{\text{T}} \tag{30-31}$$

So we get:

$$[L_{\text{s}}] = \begin{bmatrix} L_{\text{mas}} + L_{\text{las}} & -\frac{1}{2}L_{\text{mas}} & -\frac{1}{2}L_{\text{mas}} \\ -\frac{1}{2}L_{\text{mas}} & L_{\text{mas}} + L_{\text{las}} & -\frac{1}{2}L_{\text{mas}} \\ -\frac{1}{2}L_{\text{mas}} & -\frac{1}{2}L_{\text{mas}} & L_{\text{mas}} + L_{\text{las}} \end{bmatrix} = L_{\text{mas}} \cdot \begin{bmatrix} 1+\sigma_{\text{s}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\sigma_{\text{s}} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1+\sigma_{\text{s}} \end{bmatrix} \tag{32}$$

$$[L_{\text{r}}] = \begin{bmatrix} L_{\text{mar}} + L_{\text{lar}} & -\frac{1}{2}L_{\text{mar}} & -\frac{1}{2}L_{\text{mar}} \\ -\frac{1}{2}L_{\text{mar}} & L_{\text{mar}} + L_{\text{lar}} & -\frac{1}{2}L_{\text{mar}} \\ -\frac{1}{2}L_{\text{mar}} & -\frac{1}{2}L_{\text{mar}} & L_{\text{mar}} + L_{\text{lar}} \end{bmatrix} = L_{\text{mar}} \cdot \begin{bmatrix} 1+\sigma_{\text{r}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\sigma_{\text{r}} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1+\sigma_{\text{r}} \end{bmatrix} \tag{33}$$

$$[M_s] = \frac{1}{k_t} \cdot L_{mas} \cdot \begin{bmatrix} \cos\theta & \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{4\pi}{3}\right) \\ \cos\left(\theta + \frac{4\pi}{3}\right) & \cos\theta & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{4\pi}{3}\right) & \cos\theta \end{bmatrix} \tag{34}$$

$$[M_r] = k_t \cdot L_{mar} \cdot \begin{bmatrix} \cos\theta & \cos\left(\theta + \frac{4\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\theta & \cos\left(\theta + \frac{4\pi}{3}\right) \\ \cos\left(\theta + \frac{4\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\theta \end{bmatrix} \tag{35}$$

Note:

1) $[M_s] = [M_r]^T$ the mutual stator inductance matrix equals the transpose matrix of the mutual rotor inductance;

2) $\sigma_s = \frac{L_{las}}{L_{mas}} = \frac{L_{as}}{L_{mas}} - 1$; $\sigma_r = \frac{L_{lar}}{L_{mar}} = \frac{L_{ar}}{L_{mar}} - 1$ are the stator, respectively rotor leakage factors.

3) The matrix system determined above represents *the flux-current equations set* for the three-phase induction machine in a reference frame attached separately to each armature.

2.3. Space vector equations for three-phase induction machines

Considering the assumptions made in the previous paragraph, the space vector notation and concepts introduced by Racz and Kovacs (1959) are particularly useful. In this approach, all variables are represented by polar vectors indicating the magnitude and angular position for the rotating sinusoidal distribution.

A three-phase variable system can be uniquely described through a space vector, which is a complex term and time dependent $\underline{x}(t)$ and a real homopolar component $x_0(t)$:

$$\underline{x}(t) = \frac{2}{3} \cdot \left(1 \cdot x_a + \underline{\alpha} \cdot x_b + \underline{\alpha}^2 \cdot x_c\right)$$
$$x_0(t) = \frac{1}{3} \cdot \left(x_a + x_b + x_c\right) \tag{36}$$

where:

$$\underline{\alpha} = e^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$\underline{\alpha}^2 = e^{j\frac{4\pi}{3}} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

and x_a, x_b, x_c are the phase variables.

The real axis direction coincides with that one of phase a. Usually, the neutral connection for a three-phase system is open, so that the homopolar component equals zero. The phase variables can be easily obtained from the space vector notations:

$$[x_a(t) \quad x_b(t) \quad x_c(t)]^T = \text{Re}[\underline{1} \quad \underline{\alpha} \quad \underline{\alpha}^2]^T \cdot \underline{x}(t)$$

The phase voltage for the induction machine can be expressed with the help of the space vector transformation:

$$\underline{U}_s = \frac{2}{3} \cdot \left(1 \cdot U_{as} + \underline{\alpha} \cdot U_{bs} + \underline{\alpha}^2 \cdot U_{cs} \right) =$$

$$= \frac{2}{3} \cdot \left(1 \cdot I_{as} + \underline{\alpha} \cdot I_{bs} + \underline{\alpha}^2 \cdot I_{cs} \right) \cdot r_s + \frac{2}{3} \cdot \frac{d}{dt} \left(1 \cdot \lambda_{as} + \underline{\alpha} \cdot \lambda_{bs} + \underline{\alpha}^2 \cdot \lambda_{cs} \right) \tag{37}$$

or in a condensed form:

$$\underline{U}_s = r_s \underline{I}_s + \frac{d\underline{\lambda}_s}{dt} \tag{38}$$

where $\underline{U}_s, \underline{I}_s, \underline{\lambda}_s$ are space vectors for stator voltage, current and flux. Similarly, we get the rotor equation:

$$\underline{U}_r = r_r \underline{I}_r + \frac{d\underline{\lambda}_r}{dt} \tag{39}$$

where $\underline{U}_r, \underline{I}_r, \underline{\lambda}_r$ are space vectors for rotor voltage, current and flux, respectively.

Since the machine is considered magnetically linear, the stator and flux linkage will be determined as follows, making the notation:

$$\frac{2}{3} \cdot [\underline{A}] = \frac{2}{3} \cdot \begin{bmatrix} 1 & \underline{\alpha} & \underline{\alpha}^2 \end{bmatrix}$$

we get:

$$\frac{2}{3} \cdot [\underline{A}] \cdot [\underline{\lambda}_s] = \frac{2}{3} \cdot [\underline{A}] \cdot [L_s] \cdot [I_s] + \frac{2}{3} \cdot [\underline{A}] \cdot [M_s] \cdot [I_r]$$

where:

$$[\underline{A}] \cdot [L_s] = \begin{bmatrix} 1 & \underline{\alpha} & \underline{\alpha}^2 \end{bmatrix} \cdot L_{mas} \cdot \begin{bmatrix} 1+\sigma_s & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\sigma_s & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1+\sigma_s \end{bmatrix} = L_{mas} \cdot \left(\frac{3}{2} + \sigma_s \right) \cdot [\underline{A}]$$

$$[\underline{A}] \cdot [M_s] = \begin{bmatrix} 1 & \underline{\alpha} & \underline{\alpha}^2 \end{bmatrix} \cdot L_m \cdot \begin{bmatrix} \cos\theta & \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{4\pi}{3}\right) \\ \cos\left(\theta + \frac{4\pi}{3}\right) & \cos\theta & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \cos\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{4\pi}{3}\right) & \cos\theta \end{bmatrix} = \frac{3}{2} \cdot L_m \cdot [\underline{A}] \cdot e^{j\theta}$$

The condensed stator flux-current equation results from the last three equations:

$$\underline{\lambda}_s = L_{mas} \cdot \left(\frac{3}{2} + \sigma_s \right) \cdot \underline{I}_s + \frac{3}{2} \cdot L_m \cdot \underline{I}_r \cdot e^{j\theta} \tag{40}$$

where: $\underline{\lambda}_s, \underline{I}_s, \underline{I}_r$ are space vectors notations for stator flux linkage, current and rotor current.

Similarly, the rotor flux linkage-current equation is deductible:

$$\frac{2}{3} \cdot [\underline{A}] \cdot [\underline{\lambda}_r] = \frac{2}{3} \cdot [\underline{A}] \cdot [L_r] \cdot [I_r] + \frac{2}{3} \cdot [\underline{A}] \cdot [M_r] \cdot [I_s] \tag{41}$$

where:

$$[A] \cdot [L_r] = \begin{bmatrix} 1 & \underline{\alpha} & \underline{\alpha}^2 \end{bmatrix} \cdot L_m \cdot \begin{bmatrix} 1+\sigma_r & -\dfrac{1}{2} & -\dfrac{1}{2} \\[2mm] -\dfrac{1}{2} & 1+\sigma_r & -\dfrac{1}{2} \\[2mm] -\dfrac{1}{2} & -\dfrac{1}{2} & 1+\sigma_r \end{bmatrix} = L_{mar} \cdot \left(\dfrac{3}{2} + \sigma_r \right) \cdot [A]$$

$$[A] \cdot [M_r] = \begin{bmatrix} 1 & \underline{\alpha} & \underline{\alpha}^2 \end{bmatrix} \cdot L_m \cdot \begin{bmatrix} \cos\theta & \cos\left(\theta + \dfrac{4\pi}{3}\right) & \cos\left(\theta + \dfrac{2\pi}{3}\right) \\[2mm] \cos\left(\theta + \dfrac{2\pi}{3}\right) & \cos\theta & \cos\left(\theta + \dfrac{4\pi}{3}\right) \\[2mm] \cos\left(\theta + \dfrac{4\pi}{3}\right) & \cos\left(\theta + \dfrac{2\pi}{3}\right) & \cos\theta \end{bmatrix} = \dfrac{3}{2} \cdot L_m \cdot [A] \cdot e^{-j\theta}$$

Finally, we get the condensed form of the rotor flux linkage-current equation:

$$\underline{\lambda}_r = L_{mar} \cdot \left(\frac{3}{2} + \sigma_r \right) \cdot \underline{I}_r + \frac{3}{2} \cdot L_m \cdot \underline{I}_s \cdot e^{-j\theta} \tag{42}$$

where $\underline{\lambda}_r, \underline{I}_r, \underline{I}_s$ are the space vector notations for rotor flux linkage, current and stator current respectively.

For an easier manipulation of the equations we make the notations:

$$L_s = L_{mas} \cdot \left(\frac{3}{2} + \sigma_s \right)$$

$$L_r = L_{mar} \cdot \left(\frac{3}{2} + \sigma_r \right)$$

$$M = \frac{3}{2} \cdot \frac{1}{k_t} \cdot L_{mas} = \frac{3}{2} \cdot k_t \cdot L_{mar}$$

The general set of voltage and flux linkage equations, written in space vector notations, is:

$$\underline{U}_s = r_s \underline{I}_s + \frac{d\underline{\lambda}_s}{dt}$$

$$\underline{U}_r = r_r \underline{I}_r + \frac{d\underline{\lambda}_r}{dt} \tag{43-46}$$

$$\underline{\lambda}_s = L_s \underline{I}_s + M \cdot e^{j\theta} \cdot \underline{I}_r$$

$$\underline{\lambda}_r = L_r \underline{I}_r + M \cdot e^{-j\theta} \cdot \underline{I}_s$$

Some important conclusions have to be drawn regarding this mode of describing the machine equations, through space vector notations:

- The 12 scalar equations system written in natural reference frame is transformed in a 4 vector equations system. This form is equivalent to substituting the real induction machine equipped with three-phase windings on stator and rotor with a fictive machine equipped with single phase winding on stator and rotor;
- An inconvenience of the developed system is that the stator equations system is written in stator reference frame, and the rotor equations system is written in rotor reference frame, making the analysis of the machine difficult;
- The mutual inductance depends on the relative rotor position.

In Fig. 2.2 is illustrated the new fictive model of the induction machine from the space vector point of view theory.

Fig. 2.2.Model of the three-phase induction machine with fictive one stator winding and one rotor winding (space vector notations)

2.4. Vectorial equations system in a common reference frame

The analysis of an induction machine drive system has to be made when the stator and rotor variables are represented in a common reference frame. When using the same space vector notations, we can define an arbitrary reference frame, which rotates with the angular velocity ω_k, and according to Fig. 3, the following relation is valid:

$$\underline{x}^k(t) = \underline{x}(t) \cdot e^{-j\theta_k(t)} \tag{47}$$

where: $\theta_k(t)$ is the time variable relative angle between the new reference frame and the stationary reference frame initially considered.; $\underline{x}^k(t)$ represents the space vector for the new reference frame. The reverse transformation relation is:

$$\underline{x}(t) = \underline{x}^k(t) \cdot e^{j\theta_k(t)} \tag{48}$$

The homopolar component being a scalar variable, is independent from the chosen reference frame.
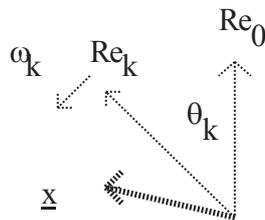
Fig. 2.3. Transformation into an arbitrary reference frame

In an arbitrary common reference frame, the vectorial voltage and flux linkage equations become:

$$\underline{U}_s = r_s \underline{I}_s + \frac{d\left(L_s \underline{I}_s + M \cdot e^{j\theta} \underline{I}_r\right)}{dt} = \underline{U}_s^k \cdot e^{j\theta_k} = r_s \underline{I}_s^k \cdot e^{j\theta_k} + \frac{d\left(L_s \underline{I}_s^k \cdot e^{j\theta_k} + M \cdot e^{j\left[(\theta_k - \theta) + \theta\right]} \underline{I}_r^k\right)}{dt} =$$

$$= r_s \underline{I}_s^k \cdot e^{j\theta_k} + L_s \cdot \left(e^{j\theta_k} \cdot \frac{d\underline{I}_s^k}{dt} + j \cdot e^{j\theta_k} \cdot \underline{I}_s^k \cdot \frac{d\theta_k}{dt}\right) + M \cdot \left(e^{j\theta_k} \cdot \frac{d\underline{I}_r^k}{dt} + j \cdot e^{j\theta_k} \cdot \underline{I}_r^k \cdot \frac{d\theta_k}{dt}\right) = \qquad (49)$$

$$= e^{j\theta_k} \cdot \left(r_s \underline{I}_s^k + \frac{d\left(L_s \underline{I}_s^k + M \underline{I}_r^k\right)}{dt} + j \cdot \omega_k \cdot \left(L_s \underline{I}_s^k + M \underline{I}_r^k\right)\right) = e^{j\theta_k} \cdot \left(r_s \underline{I}_s^k + \frac{d\underline{\lambda}_s^k}{dt} + j \cdot \omega_k \cdot \underline{\lambda}_s^k\right)$$

$$\underline{U}_r = r_r \underline{I}_r + \frac{d\left(L_r \underline{I}_r + M \cdot e^{-j\theta} \underline{I}_s\right)}{dt} = \underline{U}_r^k \cdot e^{j(\theta_k - \theta)} = r_r \underline{I}_r^k \cdot e^{j(\theta_k - \theta)} + \frac{d\left(L_r \underline{I}_r^k \cdot e^{j(\theta_k - \theta)} + M \cdot e^{j(\theta_k - \theta)} \underline{I}_s^k\right)}{dt} =$$

$$= r_r \underline{I}_r^k \cdot e^{j(\theta_k - \theta)} + L_r \cdot \left(e^{j(\theta_k - \theta)} \cdot \frac{d\underline{I}_r^k}{dt} + j \cdot e^{j(\theta_k - \theta)} \cdot \underline{I}_r^k \cdot \frac{d(\theta_k - \theta)}{dt}\right) + M \cdot \left(e^{j(\theta_k - \theta)} \cdot \frac{d\underline{I}_s^k}{dt} + j \cdot e^{j(\theta_k - \theta)} \cdot \underline{I}_s^k \cdot \frac{d(\theta_k - \theta)}{dt}\right) =$$

$$= e^{j(\theta_k - \theta)} \cdot \left(r_r \underline{I}_r^k + \frac{d\left(L_r \underline{I}_r^k + M \underline{I}_s^k\right)}{dt} + j \cdot (\omega_k - \omega) \cdot \left(L_r \underline{I}_r^k + M \underline{I}_s^k\right)\right) = e^{j(\theta_k - \theta)} \cdot \left(r_r \underline{I}_r^k + \frac{d\underline{\lambda}_r^k}{dt} + j \cdot (\omega_k - \omega) \cdot \underline{\lambda}_r^k\right)$$

$$(50)$$

where ω_k, ω are the angular velocity of the arbitrary reference frame, respectively the angular velocity of the induction machine. Finally, if we consider P the number of poles for the induction machine, the electrical angular velocity of the rotor is $(P/2)\omega$ and the vectorial equations of the induction machine written in a common arbitrary reference frame which rotates with angular velocity ω_k are:

$$\underline{U}_s^k = r_s \underline{I}_s^k + \frac{d\underline{\lambda}_s^k}{dt} + j \cdot \omega_k \cdot \underline{\lambda}_s^k$$

$$\underline{U}_r^k = r_r \underline{I}_r^k + \frac{d\underline{\lambda}_r^k}{dt} + j \cdot (\omega_k - \omega) \cdot \underline{\lambda}_r^k \qquad (51\text{-}54)$$

$$\underline{\lambda}_s^k = L_s \underline{I}_s^k + M \underline{I}_r^k$$

$$\underline{\lambda}_r^k = L_r \underline{I}_r^k + M \underline{I}_s^k$$

The above equations system represent the mathematical model of the induction machine with stator and rotor equipped with one fictive windings each in a common arbitrary reference frame. It has to be highlighted that the mutual inductance does not depend on the relative rotor position (Kovacs - 1984).
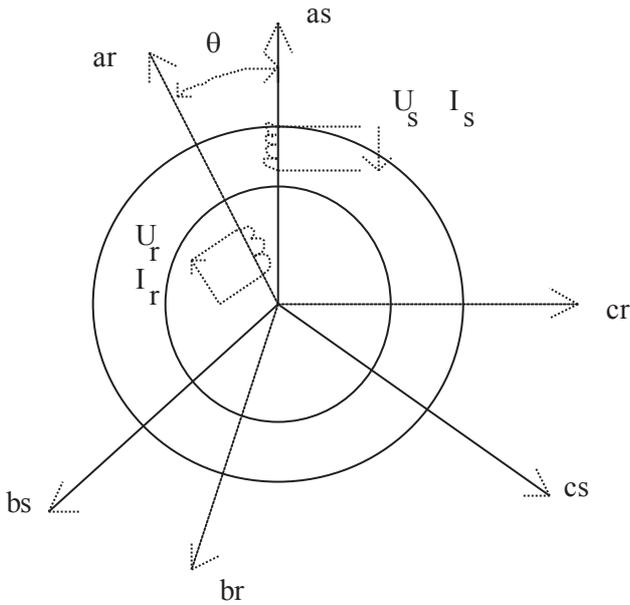

Fig. 2.4.Model of the three-phase induction machine with fictive one stator winding and one rotor winding (space vector notations) represented in a common arbitrary reference frame

2.5. Induction machine equations with stator referred rotor variables

A complete unified equations system for the induction machine is obtained when both stator and rotor variables are expressed in a common reference frame and the rotor variables are also stator referred. Using the turns ratio and the winding factor ratio we obtain:

$$\underline{L}'_r = \frac{1}{k_t} \cdot \underline{L}_r$$

$$\underline{U}'_r = k_t \cdot \underline{U}_r \quad\quad (55\text{-}58)$$

$$\underline{\lambda}'_r = k_t \cdot \underline{\lambda}_r$$

$$r'_r = k_t^2 \cdot r_r$$

Now the stator and rotor flux linkage equations can be written as:

$$\underline{\lambda}_s^k = L_{mas} \cdot \left(\frac{3}{2} + \sigma_s\right) \cdot \underline{I}_s^k + \frac{3}{2} \cdot \frac{1}{k_t} L_{mas} \cdot \underline{I}_r^k = \sigma_s \cdot L_{mas} \cdot \underline{I}_s^k + k_t \cdot M \cdot \left(\underline{I}_s^k + \underline{I}_r'^k\right) \quad\quad (59)$$

$$\underline{\lambda}_r^k = L_{mar} \cdot \left(\frac{3}{2} + \sigma_r\right) \cdot \underline{I}_r^k + \frac{3}{2} \cdot \frac{1}{k_t} L_{mar} \cdot \underline{I}_s^k = \frac{1}{k_t} \cdot \left[\sigma_r \cdot L_{mar} \cdot k_t^2 \cdot \underline{I}_r'^k + k_t \cdot M \cdot \left(\underline{I}_s^k + \underline{I}_r'^k\right)\right] \quad\quad (60)$$

If the following notations are introduced:

$L_{ls} = \sigma_s \cdot L_{mas}$ the stator leakage inductance

$L'_{lr} = k_t^2 \cdot \sigma_r \cdot L_{mar}$ the rotor leakage inductance

$L_M = k_t \cdot M = \frac{3}{2} \cdot L_{mas} = \frac{3}{2} \cdot k_t^2 \cdot L_{mar}$ the main magnetisation inductance

$\underline{I}_M^k = \underline{I}_s^k + \underline{I}_r^k$ the magnetisation current space vector

$\underline{\lambda}_M^k = L_M \cdot \underline{I}_M^k$ the magnetisation flux

$\underline{\lambda}_{ls}^k = L_{ls} \cdot \underline{I}_s^k$ the stator leakage flux

$\underline{\lambda}_{lr}'^k = L'_{lr} \cdot \underline{I}_r'^k$ the rotor leakage flux

The flux linkage equations get the following form:

$$\underline{\lambda}_s^k = L_{ls} \cdot \underline{I}_s^k + L_M \cdot \underline{I}_M^k = \underline{\lambda}_{ls}^k + \underline{\lambda}_M^k \quad\quad (61)$$

$$\underline{\lambda}_r'^k = \frac{1}{k_t} \cdot \left(L'_{lr} \cdot \underline{I}_r'^k + L_M \cdot \underline{I}_M^k\right) = \frac{1}{k_t} \cdot \left(\underline{\lambda}_{lr}'^k + \underline{\lambda}_M^k\right) \quad\quad (62)$$

or in referred variables we obtain:

$$\underline{\lambda}_s^k = (L_{ls} + L_M) \cdot \underline{I}_s^k + L_M \cdot \underline{I}_r'^k = L_s \cdot \underline{I}_s^k + L_M \cdot \underline{I}_r'^k \quad\quad (63)$$

$$\underline{\lambda}_r'^k = (L'_{lr} + L_M) \cdot \underline{I}_r'^k + L_M \cdot \underline{I}_s^k = L'_r \cdot \underline{I}_r'^k + L_M \cdot \underline{I}_s^k \quad\quad (64)$$

where L_s and L'_r are the total stator, respectively rotor self-inductance.

The referred rotor voltage is given now by the relation:

$$\underline{U}_r'^k = r'_r \underline{I}_r'^k + \frac{d\left(\underline{\lambda}_{lr}'^k + \underline{\lambda}_M^k\right)}{dt} + j \cdot \left(\omega_k - \frac{P}{2}\omega\right) \cdot \left(\underline{\lambda}_{lr}'^k + \underline{\lambda}_M^k\right) \quad\quad (65)$$

Finally we can write a complete equations system, in a common arbitrary reference frame, with rotor variables referred to the stator, which define the induction machine. The result is a mathematical model of the three-phase induction machine equipped with two fictive windings, rotating with angular velocity ω_k. The simplified representation of this system is given in Fig. 2.5.

$$\underline{U}_s^k = r_s \underline{I}_s^k + \frac{d\underline{\lambda}_s^k}{dt} + j \cdot \omega_k \cdot \underline{\lambda}_s^k$$

$$\underline{U}_r'^k = r'_r \underline{I}_r'^k + \frac{d\underline{\lambda}_r'^k}{dt} + j \cdot (\omega_k - \frac{P}{2}\omega) \cdot \underline{\lambda}_r'^k \qquad (66\text{-}69)$$

$$\underline{\lambda}_s^k = L_s \underline{I}_s^k + L_M \underline{I}_r'^k$$

$$\underline{\lambda}_r'^k = L'_r \underline{I}_r'^k + L_M \underline{I}_s^k$$
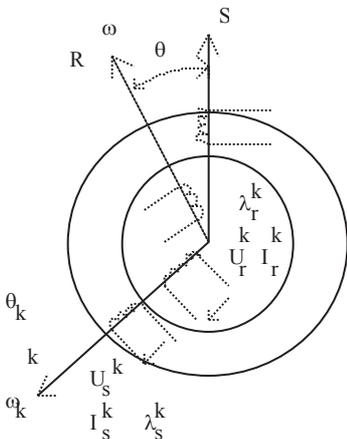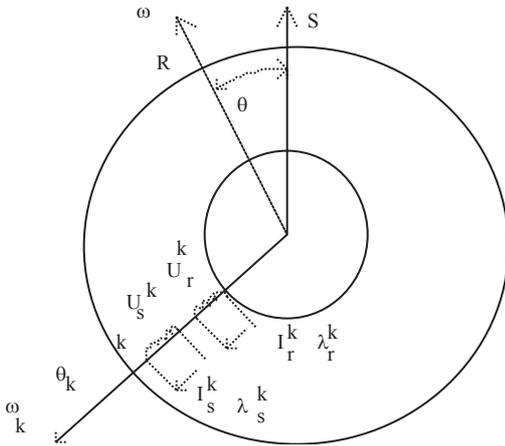

Fig. 2.5. Model of the three-phase induction machine with fictive one stator winding and one rotor winding (space vector notations) represented in a common arbitrary reference frame, and referred rotor variables

2.6. Instanteous electromagnetic torque

The input power for a three-phase induction machine with wounded rotor is:

$$P_i(t) = \frac{1}{2} \cdot m \cdot \mathrm{Re}\left\{\underline{U}_s^k \cdot \underline{I}_s^{*k}\right\} + \frac{1}{2} \cdot m \cdot \mathrm{Re}\left\{\underline{U}_r'^k \cdot \underline{I}_r'^{*k}\right\} \qquad (70)$$

where m is the phase number (3), $\underline{U}_s^k, \underline{U}_r'^k$ are the stator and rotor voltage space vectors, $\underline{I}_s^{k*}, \underline{I}_r'^{k*}$ are the conjugate stator and rotor current space vectors.
Using a detailed relation for the space vectors with rotor variables referred to the stator, we get:

$$P_i(t) = U_{as} I_{as} + U_{bs} I_{bs} + U_{cs} I_{cs} + U'_{ar} I'_{ar} + U'_{br} I'_{br} + U'_{cr} I'_{cr} \qquad (71)$$

From the complete equations system of the induction machine, the following expressions can be obtained for the input power of the machine:

$$P_i = \frac{3}{2}\mathrm{Re}\left\{r_s \underline{I}_s^k \underline{I}_s^{k*} + \frac{d\underline{\lambda}_s^k}{dt}\underline{I}_s^{k*} + j \cdot \omega_k \underline{\lambda}_s^k \underline{I}_s^{k*} + r'_r \underline{I}_r'^k \underline{I}_r'^{k*} + \frac{d\underline{\lambda}_r'^k}{dt}\underline{I}_r'^{k*} + j \cdot \left(\omega_k - \frac{P}{2}\omega\right)\underline{\lambda}_r'^k \underline{I}_r'^{k*}\right\} \qquad (72)$$

$$P_i = \frac{3}{2}\mathrm{Re}\left\{r_s I_s^2 + r'_r I_r'^2 + \frac{d\underline{\lambda}_s^k}{dt}\underline{I}_s^{k*} + \frac{d\underline{\lambda}_r'^k}{dt}\underline{I}_r'^{k*} + j \cdot \left[\omega_k \underline{\lambda}_s^k \underline{I}_s^{k*} + \left(\omega_k - \frac{P}{2}\omega\right)\underline{\lambda}_r'^k \underline{I}_r'^{k*}\right]\right\} \qquad (73)$$

The first two terms from the power equation represent the Joule effect loss, the following two terms represent the electromagnetic power due to the time variation of the magnetic energy, and the last term stands for the mechanical power available at the machine shaft, if the hysterezis loss, eddy current loss and the stray losses are neglected. The mechanical power will be:

$$P_{\mathrm{m}} = \frac{3}{2}\mathrm{Re}\left\{ j\cdot\left[\omega_k\left(L_{\mathrm{s}}\underline{I}_{\mathrm{s}}^k + L_{\mathrm{M}}\underline{I}_{\mathrm{r}}^{\prime k*}\right)\cdot\underline{I}_{\mathrm{s}}^{k*} + \left(\omega_k - \frac{P}{2}\omega\right)\cdot\left(L_{\mathrm{r}}\underline{I}_{\mathrm{r}}^{\prime k} + L_{\mathrm{M}}\underline{I}_{\mathrm{s}}^k\right)\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right]\right\} =$$

$$= \frac{3}{2}\mathrm{Re}\left\{-j\cdot\frac{P}{2}\cdot\omega\cdot L_{\mathrm{M}}\cdot\underline{I}_{\mathrm{s}}^k\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} = \frac{3P}{4}\cdot\omega\cdot L_{\mathrm{M}}\cdot\mathrm{Im}\left\{\underline{I}_{\mathrm{s}}^k\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\}$$

(74)

Using the flux linkage relations, three expressions are deductible for the induction machine mechanical power:

$$P_{\mathrm{m}} = \frac{3P}{4}\cdot\omega\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{r}}^{\prime k}\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} = \frac{3P}{4}\cdot\omega\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{s}}^{k*}\cdot\underline{I}_{\mathrm{s}}^k\right\} = \frac{3P}{4}\cdot\omega\cdot L_{\mathrm{M}}\cdot\mathrm{Im}\left\{\underline{I}_{\mathrm{s}}^k\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} =$$

$$= \frac{3P}{4}\cdot\omega\cdot\frac{L_{\mathrm{M}}}{L_{\mathrm{r}}'}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{r}}^{\prime k*}\cdot\underline{I}_{\mathrm{s}}^k\right\} = \frac{3P}{4}\cdot\omega\cdot\frac{L_{\mathrm{M}}}{\Delta}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{s}}^{k*}\cdot\underline{\lambda}_{\mathrm{r}}^{\prime k}\right\}$$

(75)

where:

$$\Delta = L_{\mathrm{s}}L_{\mathrm{r}}' - L_{\mathrm{M}}^2$$

which gives four expressions for the instantaneous electromagnetic torque if the mechanical power is divided by the rotor angular velocity:

$$T_{\mathrm{e}} = \frac{3P}{4}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{r}}^{\prime k}\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} = \frac{3P}{4}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{s}}^{k*}\cdot\underline{I}_{\mathrm{s}}^k\right\} = \frac{3P}{4}\cdot L_{\mathrm{M}}\cdot\mathrm{Im}\left\{\underline{I}_{\mathrm{s}}^k\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} =$$

$$= \frac{3P}{4}\cdot\frac{L_{\mathrm{M}}}{L_{\mathrm{r}}'}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{r}}^{\prime k*}\cdot\underline{I}_{\mathrm{s}}^k\right\} = \frac{3P}{4}\cdot\frac{L_{\mathrm{M}}}{\Delta}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{s}}^{k*}\cdot\underline{\lambda}_{\mathrm{r}}^{\prime k}\right\}$$

(76)

The torque and the rotor speed a related by the mechanical equation:

$$T_{\mathrm{e}} = J\cdot\left(\frac{2}{P}\right)\frac{d\omega}{dt} + T_{\mathrm{L}}$$

(77)

where: J is the inertia of the rotor and in some cases the connected load. The first term on the right-hand side of the equation is the inertial torque. The load torque T_{L} is positive for a torque load on the shaft of the induction machine.

Considering the voltage, flux linkage and mechanical equations, the complete relations system of the induction machine can be written:

$$\underline{U}_{\mathrm{s}}^k = r_{\mathrm{s}}\underline{I}_{\mathrm{s}}^k + \frac{d\underline{\lambda}_{\mathrm{s}}^k}{dt} + j\cdot\omega_k\cdot\underline{\lambda}_{\mathrm{s}}^k$$

$$\underline{U}_{\mathrm{r}}^{\prime k} = r_{\mathrm{r}}'\underline{I}_{\mathrm{r}}^{\prime k} + \frac{d\underline{\lambda}_{\mathrm{r}}^{\prime k}}{dt} + j\cdot\left(\omega_k - \frac{P}{2}\omega\right)\cdot\underline{\lambda}_{\mathrm{r}}^{\prime k}$$

$$\underline{\lambda}_{\mathrm{s}}^k = L_{\mathrm{s}}\underline{I}_{\mathrm{s}}^k + L_{\mathrm{M}}\underline{I}_{\mathrm{r}}^{\prime k}$$

(78-82)

$$\underline{\lambda}_{\mathrm{r}}^{\prime k} = L_{\mathrm{r}}'\underline{I}_{\mathrm{r}}^{\prime k} + L_{\mathrm{M}}\underline{I}_{\mathrm{s}}^k$$

$$T_{\mathrm{e}} = \frac{3P}{4}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{r}}^{\prime k}\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} = \frac{3P}{4}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{s}}^{k*}\cdot\underline{I}_{\mathrm{s}}^k\right\} = \frac{3P}{4}\cdot L_{\mathrm{M}}\cdot\mathrm{Im}\left\{\underline{I}_{\mathrm{s}}^k\cdot\underline{I}_{\mathrm{r}}^{\prime k*}\right\} =$$

$$= \frac{3P}{4}\cdot\frac{L_{\mathrm{M}}}{L_{\mathrm{r}}'}\cdot\mathrm{Im}\left\{\underline{\lambda}_{\mathrm{r}}^{\prime k*}\cdot\underline{I}_{\mathrm{s}}^k\right\} = J\cdot\left(\frac{2}{P}\right)\frac{d\omega}{dt} + T_{\mathrm{L}}$$

The above set of equations is valid for the following conditions:

- All the equations are written in a common reference frame, which rotates with the arbitrary angular velocity ω_k.
- The rotor variables are referred to the stator;
- The stator and rotor variables are described by the space vector notations;
- The machine parameters are constant in a common reference frame for both armatures;
- The mechanical equation is written in real domain, and is independent of the chosen reference frame.

2.7. General equations of the induction machine in different reference frames

When the mathematical model of the induction machine is established, several reference frames can be employed depending on application and the chosen strategy control. There are several main reference frames: stationary fixed to the stator, synchronously fixed to the rotor, and revolving with an angular velocity equal to: the air-gap flux, the rotor flux, the stator voltage, the rotor current space vectors. The transformation from one reference frame to another is made by keeping constant the value of the m.m.f. as all the reference frames are energetically equivalent. For the induction machine, the phasorial diagram of the main space vectors is illustrated in Fig. 2.6. The significance of the index is as follows:

s – stationary reference frame linked to phase *as* of the stator windings
r – rotating reference frame linked to the rotor shaft
m – air-gap flux reference frame
ss – stator total synchronous reference frame
rs – rotor flux synchronous reference frame
k – arbitrary synchronous reference frame
The angles depicted in Fig. 2.6 are in electrical degrees.
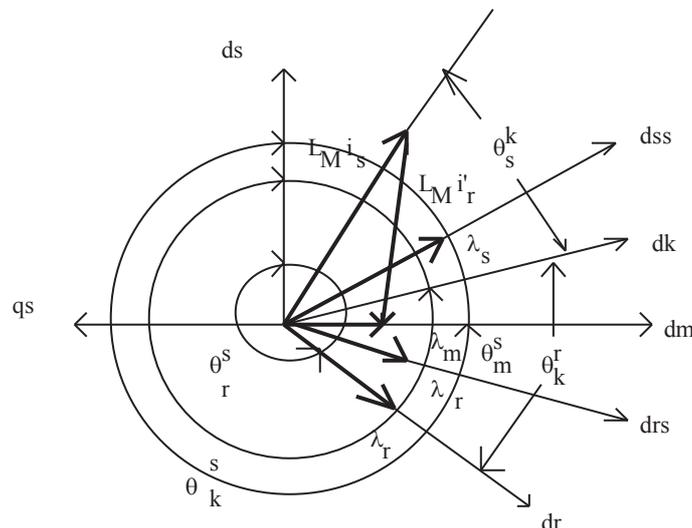
Fig. 2.6. Definitions of the electrical angles between different reference frames

2.7.1. Per unit system

It is often convenient to express machine parameters and variables as per unit quantities. The mathematical model in per unit representation of the induction machine has some main features (Krause et al. 1995):

- All parameters have maximum value equal to unit;
- Two similar systems can be compared easier;
- The digital control is readily implemented;

When using the per unit system for writing the machine equations, the following observations have to be made:

a) The base torque is not the rated one. As the rated power output generally occurs at a speed (rated speed) slightly less than synchronous, the base torque T_b will be less than rated torque by the ratio of rated speed to synchronous speed of the machine.

b) In per unit system, the inductance value is equal to the reactance value.

c) If the flux linkage per second is used in the induction machine equations, it is per unitised by dividing by base voltage.

The following values of the machine parameters are used as base variables:

$U_b = U_{sn,max} = \sqrt{2} \cdot U_{s(rms),n}$	-phase voltage, maximum value
$I_b = I_{sn.max} = \sqrt{2} \cdot I_{s(rms),n}$	-phase current, maximum value
$Z_b = \dfrac{U_b}{I_b}$	-base impedance
$S_b = \dfrac{3}{2} U_b I_b = 3 U_{s(rms),n} I_{s(rms)n}$	-base power
$\omega_{sb} = \omega_{sn} = 2\pi \cdot f_n$	-base stator variables angular velocity
$\omega_b = \dfrac{2}{P} \cdot \omega_{sb} = \dfrac{2}{P} \cdot \omega_{sn} = \dfrac{4\pi}{P} \cdot f_n$	-base rotor variables angular velocity
$\lambda_b = \dfrac{U_b}{\omega_{sb}}$	-base stator flux
$L_b = \dfrac{\lambda_b}{I_b}$	-base inductance
$T_b = \dfrac{S_b}{\omega_b} = \dfrac{3P}{4} \cdot \dfrac{U_{sn.max} I_{sn,max}}{\omega_{sn}}$	-base torque
$H = \dfrac{1}{2} \cdot \dfrac{J \omega_b^2}{S_b}$	-inertia constant

The stator voltage equation in per unit system becomes:

$$\frac{\underline{U}_s^k}{U_b} = \frac{r_s \underline{I}_s^k I_b}{U_b I_b} + \frac{d \underline{\lambda}_s^k}{dt} \cdot \frac{1}{U_b} + \frac{j \cdot \omega_k \underline{\lambda}_s^k}{U_b}$$

$$\underline{u}_s^k = R_s \underline{i}_s^k + \frac{1}{\omega_{sn}} \cdot \frac{d \underline{\psi}_s^k}{dt} + j \cdot \Omega_k \underline{\psi}_s^k$$

(83-84)

Similarly we obtain the rotor voltage equation:

$$\underline{u}_r'^k = R'_r \underline{i}_r'^k + \frac{1}{\omega_{sn}} \cdot \frac{d \underline{\psi}_r'^k}{dt} + j \cdot (\Omega_k - \Omega) \cdot \underline{\psi}_r'^k \tag{85}$$

where the relative angular velocity is: $\Omega = \dfrac{\omega}{\omega_b}$

In the previous per unit expressions we should note that the multiplying coefficient of the time flux derivative is necessary, as the time is not referred. The relative time, defined as follows, can be used:

$$t_{relative} = \omega_{sb} t = \omega_{sn} t = 2\pi \cdot \frac{t}{T_{sn}} \tag{86}$$

where T_{sn} represent the stator voltage supply period.

Finally the complete equations set in per unit system for the induction machine is:

$$\underline{u}_s^k = R_s \underline{i}_s^k + \frac{1}{\omega_{sb}} \cdot \frac{d\underline{\psi}_s^k}{dt} + j \cdot \Omega_k \underline{\psi}_s^k$$

$$\underline{u}_r^{\prime k} = R_r^{\prime} \underline{i}_r^{\prime k} + \frac{1}{\omega_{sb}} \cdot \frac{d\underline{\psi}_r^{\prime k}}{dt} + j \cdot (\Omega_k - \Omega) \cdot \underline{\psi}_r^{\prime k}$$

$$\underline{\psi}_s^k = x_s \underline{i}_s^k + x_M \underline{i}_r^{\prime k} \qquad\qquad (87\text{-}91)$$

$$\underline{\psi}_r^{\prime k} = x_r^{\prime} \underline{i}_r^{\prime k} + x_M \underline{i}_s^k$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \operatorname{Im}\left\{ \underline{\psi}_s^{k*} \cdot \underline{i}_s^k \right\} - T_L \right)$$

2.7.2. Stationary reference frame equations. Block diagram

If the stator voltages are unbalanced or discontinuous and the rotor-applied voltages are balanced or zero, the most appropriate choice for the reference frame is the one fixed to the stator. This stationary reference frame was first employed by Stanley (1938). In a stationary reference frame, fixed to the stator, the arbitrary angular velocity is zero, ($\omega_k = 0$) and the induction machine equations system becomes:

$$\underline{u}_s^k = R_s \underline{i}_s^k + \frac{1}{\omega_{sb}} \cdot \frac{d\underline{\psi}_s^k}{dt}$$

$$\underline{u}_r^{\prime k} = R_r^{\prime} \underline{i}_r^{\prime k} + \frac{1}{\omega_{sb}} \cdot \frac{d\underline{\psi}_r^{\prime k}}{dt} - j \cdot \Omega \cdot \underline{\psi}_r^{\prime k}$$

$$\underline{\psi}_s^k = x_s \underline{i}_s^k + x_M \underline{i}_r^{\prime k} \qquad\qquad (92\text{-}96)$$

$$\underline{\psi}_r^{\prime k} = x_r^{\prime} \underline{i}_r^{\prime k} + x_M \underline{i}_s^k$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \operatorname{Im}\left\{ \underline{\psi}_s^{k*} \cdot \underline{i}_s^k \right\} - T_L \right)$$

The total leakage factor is described by relation $\sigma_t = 1 - \dfrac{x_M^2}{x_s x_r}$ and if we note: $\delta = x_s x_r - x_M^2$ the following relations are deductible:

$$\underline{i}_s^k = \frac{x_r}{\delta} \underline{\psi}_s^k - \frac{x_M}{\delta} \underline{\psi}_r^{\prime k}$$

$$\qquad\qquad (97\text{-}98)$$

$$\underline{i}_r^{\prime k} = \frac{x_s}{\delta} \underline{\psi}_r^{\prime k} - \frac{x_M}{\delta} \underline{\psi}_s^k$$

A new equation system can be written in the two-axis co-ordinates system:

$$u_{qs} = R_s i_{qs} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qs}}{dt}$$

$$u_{ds} = R_s i_{ds} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{ds}}{dt}$$

$$u'_{qr} = R'_r i'_{qr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi'_{qr}}{dt} - \Omega \cdot \psi'_{dr}$$

$$u'_{dr} = R'_r i'_{dr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi'_{dr}}{dt} + \Omega \cdot \psi'_{qr}$$

$$\psi_{qs} = x_s i_{qs} + x_M i'_{qr}$$ (99-107)

$$\psi_{ds} = x_s i_{ds} + x_M i'_{dr}$$

$$\psi'_{qr} = x'_r i'_{qr} + x_M i_{qs}$$

$$\psi'_{dr} = x'_r i'_{dr} + x_M i_{ds}$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds} \right) - T_L \right)$$

If the induction machine is equipped with cage rotor, then the rotor voltage is zero. A complete block diagram for the induction machine in stationary reference frame using as inputs stator voltage and currents and load torque, and as output the speed, is illustrated in Fig. 2.7.

Fig. 2.7. Block diagram of the induction machine in stationary reference frame

2.7.3. Rotor reference frame equations. Block diagram

The choice of the reference frame for the dynamic analysis of the induction machine, especially when the rotor circuits are unbalanced, is more convenient to be fixed to the rotor frame. This reference frame is in fact the Park's transformation, initially developed for synchronous machine and than applied to the induction machine by Brereton (Fitzgerald et al -1990). The method of referring the machine variables to a rotor reference frame is most useful for field oriented control

systems. Transformation from the arbitrary frame to the rotor frame is made by substituting $\Omega_k = \Omega$, or from the stationary frame with the matrix relations:

$$
\begin{bmatrix} x_{dr} \\ x_{qr} \end{bmatrix} = \begin{bmatrix} \cos\theta_r & \sin\theta_r \\ -\sin\theta_r & \cos\theta_r \end{bmatrix} \cdot \begin{bmatrix} x_{ds} \\ x_{qs} \end{bmatrix}
$$

$$
\begin{bmatrix} x_{ds} \\ x_{qs} \end{bmatrix} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix} \cdot \begin{bmatrix} x_{dr} \\ x_{qr} \end{bmatrix}
$$

(108-109)

where θ_r is the electrical angle between the magnetic axis of the stator flux and the magnetic axis of the rotor flux.

The complete equations set results as follows:

$$
\underline{u}_s^k = R_s \underline{i}_s^k + \frac{1}{\omega_{sb}} \cdot \frac{d\underline{\psi}_s^k}{dt} + j \cdot \Omega \cdot \underline{\psi}_s^k
$$

$$
\underline{u}'^k_r = R'_r \, \underline{i}'^k_r + \frac{1}{\omega_{sb}} \cdot \frac{d\underline{\psi}'^k_r}{dt}
$$

$$
\underline{\psi}_s^k = x_s \underline{i}_s^k + x_M \underline{i}'^k_r
$$

(110-114)

$$
\underline{\psi}'^k_r = x'_r \, \underline{i}'^k_r + x_M \, \underline{i}_s^k
$$

$$
\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \mathrm{Im} \left\{ \underline{\psi}'^k_r \cdot \underline{i}'^{k*}_r \right\} - T_L \right)
$$

Or expressed in orthogonal two-axis co-ordinate system:

$$
u_{qs} = R_s i_{qs} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qs}}{dt} + \Omega\psi_{ds}
$$

$$
u_{ds} = R_s i_{ds} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{ds}}{dt} - \Omega\psi_{qs}
$$

$$
u'_{qr} = R'_r i'_{qr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi'_{qr}}{dt}
$$

$$
u'_{dr} = R'_r i'_{dr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi'_{dr}}{dt}
$$

(115-123)

$$
\psi_{qs} = x_s i_{qs} + x_M i'_{qr}
$$

$$
\psi_{ds} = x_s i_{ds} + x_M i'_{dr}
$$

$$
\psi'_{qr} = x'_r i'_{qr} + x_M i_{qs}
$$

$$
\psi'_{dr} = x'_r i'_{dr} + x_M i_{ds}
$$

$$
\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \left(\psi'_{qr} i'_{dr} - \psi'_{dr} i'_{qr} \right) - T_L \right)
$$

The block diagram of the induction machine model in rotor reference frame is presented in Fig. 2.8. The input variables are stator currents and load torque, and as output variable is chosen the angular velocity of the rotor. Also, the rotor voltages are considered zero (case of squirrel-cage rotor). It has to be observed that two algebraic loops appear in this diagram. This inconvenient leads to a limited use of this reference frame to the wound rotor induction machine, when the block diagram is similar to that from Fig. 2.8, where inputs will be rotor voltages and currents.
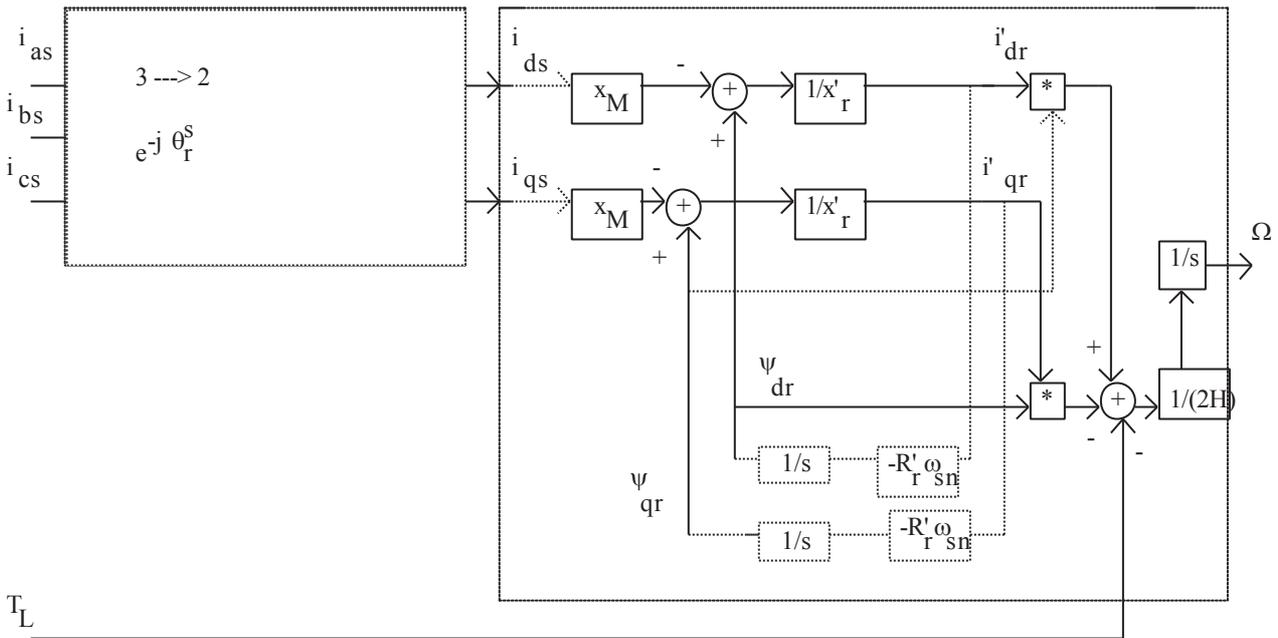
Fig.2.8. Block diagram of the induction machine in rotor reference frame (cage rotor case)
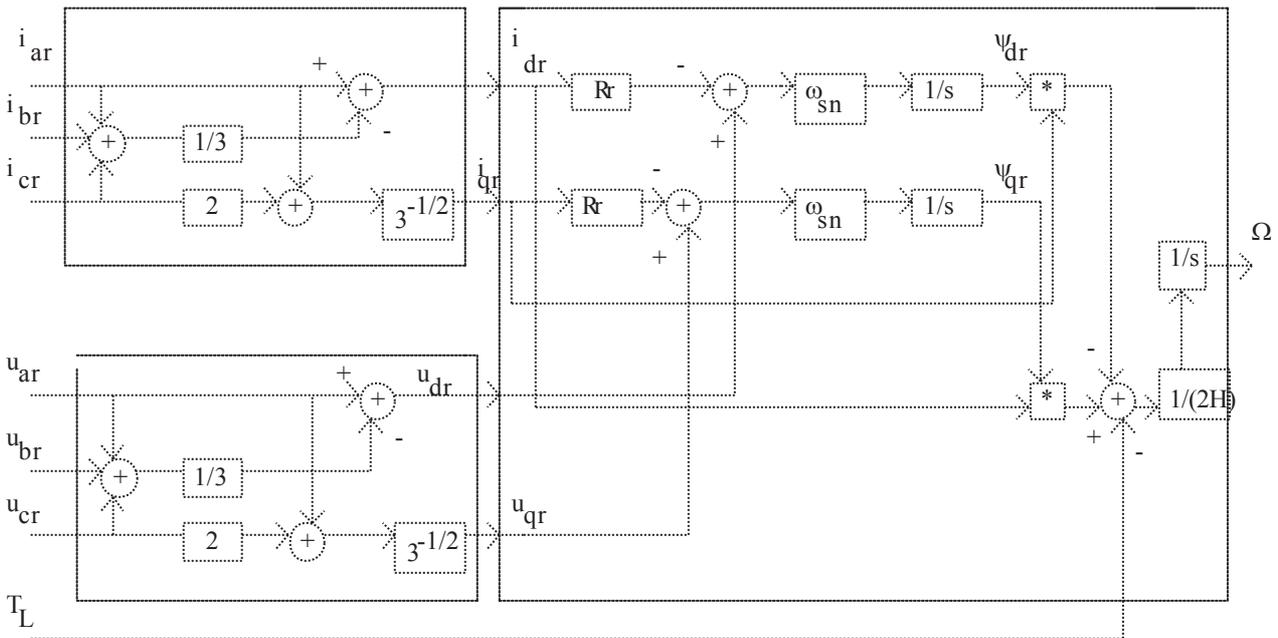
Fig. 2.9. Block diagram of the induction machine in rotor reference frame (wound rotor case)

2.7.4. Synchronous reference frame equations. Block diagram

The synchronously rotating reference frame, with angular velocity equal to that one of the power supply system, is particularly convenient when incorporating the dynamic characteristics of an induction machine into a digital computer program used to study the transient and dynamic stability of power systems. The synchronously rotating reference frame may also be useful in variable frequency applications if we may assume that the stator voltages are a sinusoidal balanced set. It was systematically developed by Kovacs (1984) and Krause et al (1995), or Lorenz et al. (1994).

When a two-axis co-ordinates reference frame is employed, it has to be fixed to different variables of the machine. The main configurations are the synchronously reference frame fixed to: stator flux, air-gap flux and rotor flux.

I) Stator flux fixed synchronous reference frame.

In order to link the *d*-axis of the synchronous reference frame to the stator flux space vector, the *q*-component of this flux vector is defined equal to zero:

$$\underline{\psi}_s^s = \psi_{ds}^s$$
$$\psi_{qs}^s = 0$$

(124)

The following equations set is obtainable:

$$u_{qs}^s = R_s i_{qs}^s + \Omega_s \psi_{ds}^s$$

$$u_{ds}^s = R_s i_{ds}^s + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{ds}^s}{dt}$$

$$u_{qr}^{'s} = R_r' i_{qr}^{'s} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qr}^{'s}}{dt} + (\Omega_s - \Omega) \cdot \psi_{dr}^{'s}$$

$$u_{dr}^{'s} = R_r' i_{dr}^{'s} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{dr}^{'s}}{dt} - (\Omega_s - \Omega) \cdot \psi_{qr}^{'s}$$

$$0 = x_s i_{qs}^s + x_M i_{qr}^{'s}$$

$$\psi_{ds}^s = x_s i_{ds}^s + x_M i_{dr}^{'s}$$

$$\psi_{qr}^{'s} = x_r' i_{qr}^{'s} + x_M i_{qs}^s$$

$$\psi_{dr}^{'s} = x_r' i_{dr}^{'s} + x_M i_{ds}^s$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \psi_{ds}^s i_{qs}^s - T_L \right)$$

(125-133)

The block diagram from Fig. 2.10 presents the induction machine model in synchronous reference frame linked to the stator flux, with inputs stator voltages and currents, and load torque. The outputs are synchronous and rotor angular velocity.
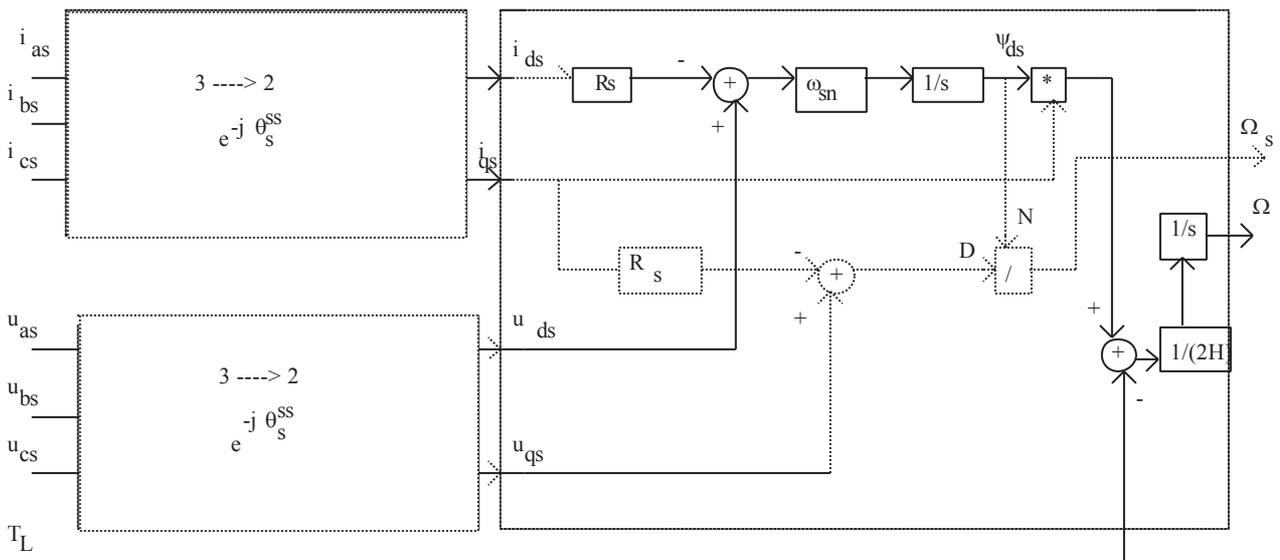
Fig. 2.10. Block diagram of the induction machine in synchronous reference frame
(linked to stator flux)

II) Rotor flux fixed synchronous reference frame.

In order to link the *d*-axis of the synchronous reference frame to the rotor flux space vector, the *q*-component of this flux vector is defined equal to zero:

$$\underline{\psi}^{\text{rr}}_{r} = \psi'^{\text{rr}}_{dr}$$

$$\psi'^{\text{rr}}_{qr} = 0 \tag{134}$$

The induction machine characteristics can be analysed with the equation system:

$$u^{\text{r}}_{qs} = R_s i^{\text{r}}_{qs} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi^{\text{r}}_{qs}}{dt} + \Omega_s \psi^{\text{r}}_{ds}$$

$$u^{\text{r}}_{ds} = R_s i^{\text{r}}_{ds} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi^{\text{r}}_{ds}}{dt} - \Omega_s \psi^{\text{r}}_{qs}$$

$$u'^{\text{r}}_{qr} = 0 = R'_r i'^{\text{r}}_{qr} + (\Omega_s - \Omega) \cdot \psi''^{\text{r}}_{dr}$$

$$u'^{\text{r}}_{dr} = 0 = R'_r i'^{\text{r}}_{dr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi''^{\text{r}}_{dr}}{dt}$$

$$\psi^{\text{r}}_{qs} = x_s i^{\text{r}}_{qs} + x_M i'^{\text{r}}_{qr} \tag{135-143}$$

$$\psi^{\text{r}}_{ds} = x_s i^{\text{r}}_{ds} + x_M i'^{\text{r}}_{dr}$$

$$0 = x'_r i'^{\text{r}}_{qr} + x_M i^{\text{r}}_{qs}$$

$$\psi''^{\text{r}}_{dr} = x'_r i'^{\text{r}}_{dr} + x_M i^{\text{r}}_{ds}$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(-\frac{3P}{4} \psi''^{\text{r}}_{dr} i'^{\text{r}}_{qr} - T_L \right) = \frac{1}{2H} \cdot \left(\frac{3P}{4} \cdot \frac{L_M}{L'_r} \psi''^{\text{r}}_{dr} i^{\text{r}}_{qs} - T_L \right)$$

The block diagram from Fig. 2.11 presents the induction machine model in synchronous reference frame linked to the rotor flux, with inputs stator currents, and load torque. The outputs are synchronous and rotor angular velocity. The rotor voltages are considered equal to zero (case of the cage rotor).

Fig. 2.11. Block diagram of the induction machine in synchronous reference frame
(linked to rotor flux)

III) Air-gap flux fixed synchronous reference frame.

In order to link the *d*-axis of the synchronous reference frame to the air-gap flux space vector, the *q*-component of this flux vector is defined equal to zero:

$$\underline{\psi}_m^m = \psi_{dm}^m$$
$$\psi_{qm}^m = 0 \tag{144}$$

The system is described by the relations:

$$u_{qs}^m = R_s i_{qs}^m + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qs}^{\,m}}{dt} + \Omega_s \psi_{ds}^{\,m}$$

$$u_{ds}^m = R_s i_{ds}^m + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{ds}^{\,m}}{dt} - \Omega_s \psi_{qs}^{\,m}$$

$$u_{qr}^{\prime m} = 0 = R_r^{\prime} \, i_{qr}^{\prime m} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qr}^{\prime\,m}}{dt} + (\Omega_s - \Omega) \cdot \psi_{dr}^{\prime\,m}$$

$$u_{dr}^{\prime m} = 0 = R_r^{\prime} \, i_{dr}^{\prime m} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{dr}^{\prime\,m}}{dt} - (\Omega_s - \Omega) \cdot \psi_{qr}^{\prime\,m}$$

$$0 = x_M (i_{qs}^m + i_{qr}^{\prime m})$$

$$\psi_{dm}^{\,m} = x_M (i_{ds}^m + i_{dr}^{\prime\,m}) \tag{145-155}$$

$$\psi_{ds}^{\,m} = x_s i_{ds}^m + x_M i_{dr}^{\prime\,m} = \psi_{dm}^{\,m} + x_{ls} i_{ds}^m$$

$$\psi_{qs}^{\,m} = x_s i_{qs}^m + x_M i_{qr}^{\prime\,m} = x_{ls} i_{qs}^m$$

$$\psi_{qr}^{\prime\,m} = x_r^{\prime} \, i_{qr}^{\prime m} + x_M i_{qs}^m = x_{lr}^{\prime} \, i_{qr}^{\prime m}$$

$$\psi_{dr}^{\prime\,m} = x_r^{\prime} i_{dr}^{\prime\,m} + x_M i_{ds}^m = \psi_{dm}^{\,m} + x_{lr}^{\prime} i_{dr}^{\prime\,m}$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \psi_{dm}^{\,m} i_{qs}^m - T_L \right)$$

Fig. 2.12 illustrates the induction machine model in synchronous reference frame linked to the air-gap flux, with inputs stator voltages and currents, synchronous angular velocity and load torque. The output is rotor angular velocity.

Fig. 2.12. Block diagram of the induction machine in synchronous reference frame
(linked to air-gap flux)

IV) Stator current space vector fixed synchronous reference frame.

In order to link the *d*-axis of the synchronous reference frame to the stator current space vector, the *q*-component of this space vector is defined equal to zero:

$$\underline{i}_s^{sc} = i_{ds}^{sc}$$
$$i_{qs}^{sc} = 0 \tag{156}$$

This reference frame realises a special control of the machine torque and speed, by using only one current component and the stator voltage. De-coupling circuits are needed for an independent control of the speed and torque. The complete set of equations which describes the dynamic and steady-state operation of the induction machine is.

$$u_{qs}^{sc} = \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qs}^{sc}}{dt} + \Omega_s \psi_{ds}^{sc}$$

$$u_{ds}^{sc} = R_s i_{ds}^{sc} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{ds}^{sc}}{dt} - \Omega_s \psi_{qs}^{sc}$$

$$u_{qr}'^{sc} = 0 = R_r' i_{qr}'^{sc} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{qr}'^{sc}}{dt} + (\Omega_s - \Omega) \cdot \psi_{dr}'^{sc}$$

$$u_{dr}'^{sc} = 0 = R_r' i_{dr}'^{sc} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi_{dr}'^{sc}}{dt} - (\Omega_s - \Omega) \cdot \psi_{qr}'^{sc}$$

$$\psi_{ds}^{sc} = x_s i_{ds}^{sc} + x_M i_{dr}'^{sc} \tag{157-165}$$

$$\psi_{qs}^{sc} = x_M i_{qr}'^{sc}$$

$$\psi_{qr}'^{sc} = x_r' i_{qr}'^{sc}$$

$$\psi_{dr}'^{sc} = x_r' i_{dr}'^{sc} + x_M i_{ds}^{sc}$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(-\frac{3P}{4} \psi_{qs}^{sc} \cdot i_{sd}^{sc} - T_L \right)$$

The block diagram from Fig. 2.13, uses as inputs the stator voltage space vector, the d-axis stator current component, the load torque, and the synchronous angular velocity. As output is considered the rotor angular velocity (speed).

Fig. 2.13. Block diagram of the induction machine in synchronous reference frame
(linked to stator current space vector)

V) Rotor current space vector fixed synchronous reference frame.

In order to link the *d*-axis of the synchronous reference frame to the rotor current space vector, the *q*-component of this space vector is defined equal to zero:

$$\underline{i}\,'^{rc}_{r} = i\,'^{rc}_{dr}$$

$$i^{rc}_{qr} = 0 \tag{166}$$

Applying the above conditions, a complete equations system for the induction machine expressed in this particularly reference frame can be deduced:

$$u^{rc}_{qs} = \frac{1}{\omega_{sb}} \cdot \frac{d\psi^{rc}_{qs}}{dt} + \Omega_s \psi^{rc}_{ds}$$

$$u^{rc}_{ds} = R_s i^{rc}_{ds} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi^{rc}_{ds}}{dt} - \Omega_s \psi^{rc}_{qs}$$

$$u'^{rc}_{qr} = 0 = \frac{1}{\omega_{sb}} \cdot \frac{d\psi\,'^{rc}_{qr}}{dt} + (\Omega_s - \Omega) \cdot \psi\,'^{rc}_{dr}$$

$$u'^{sc}_{dr} = 0 = R'_r i\,'^{rc}_{dr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi\,'^{rc}_{dr}}{dt} - (\Omega_s - \Omega) \cdot \psi\,'^{rc}_{qr}$$

$$\psi^{rc}_{ds} = x_s i^{rc}_{ds} + x_M i\,'^{rc}_{dr} \tag{167-175}$$

$$\psi^{rc}_{qs} = x_s i^{rc}_{qs}$$

$$\psi\,'^{rc}_{qr} = x_M i^{rc}_{qs}$$

$$\psi\,'^{rc}_{dr} = x'_r i\,'^{rc}_{dr} + x_M i^{rc}_{ds}$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} \psi\,'^{rc}_{qr} \cdot i'^{rc}_{dr} - T_L \right) = \frac{1}{2H} \cdot \left(\frac{3P}{4} x_M \cdot i^{rc}_{qs} \cdot i'^{rc}_{dr} - T_L \right)$$

In Fig. 2.14, the block diagram that describes the induction machine model has as inputs the d-axis stator voltage component and current space vector, the load torque, and the synchronous angular velocity. As output is considered the rotor angular velocity (speed).
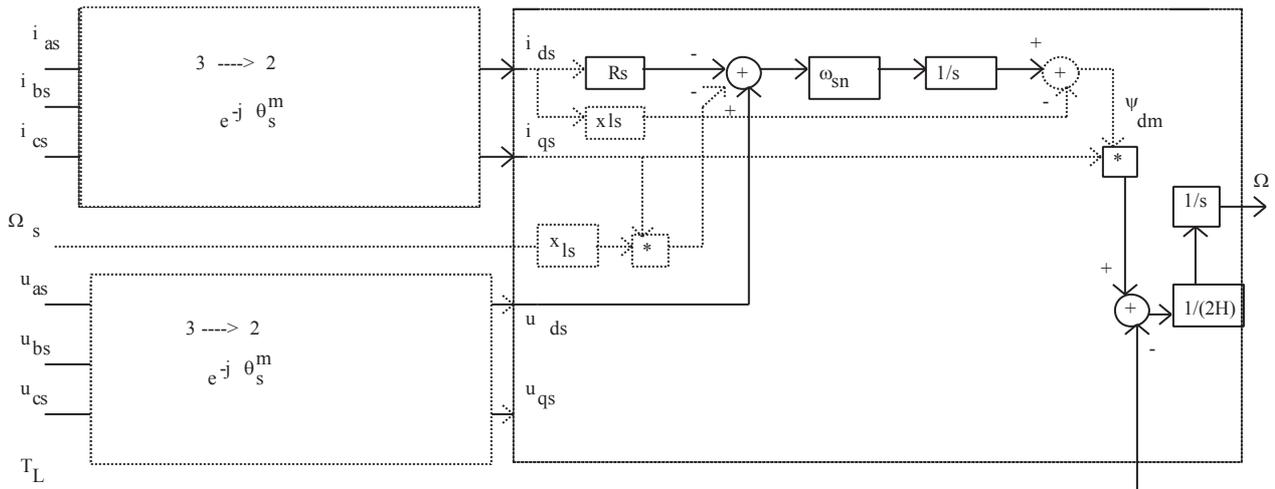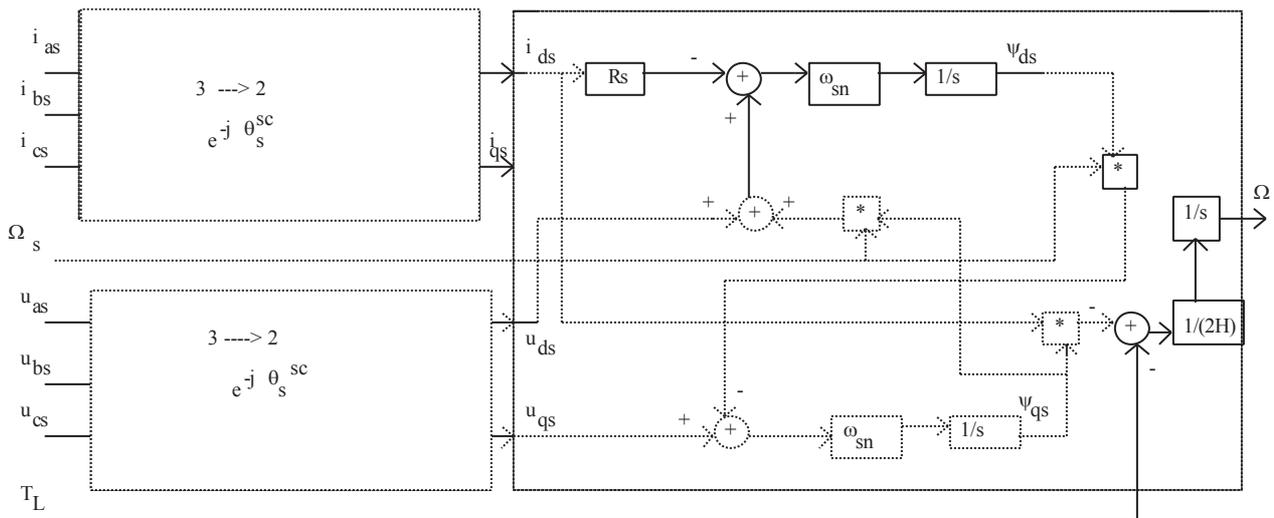

Fig. 2.14. Block diagram of the induction machine in synchronous reference frame
(linked to rotor current space vector)

VI) Magnetising current space vector fixed synchronous reference frame.

In order to link the *d*-axis of the synchronous reference frame to the magnetising current space vector, the *q*-component of this space vector is defined equal to zero:

$$\underline{i}_m^{mc} = i_{dm}^{mc} = \underline{i}_s^{mc} + \underline{i}\,'_r^{mc}$$
$$i_{qm}^{mc} = 0 \tag{176}$$

Considering the above conditions for a synchronous reference frame, the complete set of equation for the three-phase induction machine can be obtained:

$$u^{mc}_{qs} = R_s i^{mc}_{qs} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi^{mc}_{qs}}{dt} + \Omega_s \psi^{mc}_{ds}$$

$$u^{mc}_{ds} = R_s i^{mc}_{ds} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi^{mc}_{ds}}{dt} - \Omega_s \psi^{mc}_{qs}$$

$$u\,'^{mc}_{qr} = 0 = R\,'_r i\,'^{mc}_{qr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi\,'^{mc}_{qr}}{dt} + (\Omega_s - \Omega) \cdot \psi\,'^{mc}_{dr}$$

$$u\,'^{mc}_{dr} = 0 = R\,'_r i^{mc}_{dr} + \frac{1}{\omega_{sb}} \cdot \frac{d\psi\,'^{mc}_{dr}}{dt} - (\Omega_s - \Omega) \cdot \psi\,'^{mc}_{qr}$$

$$i^{mc}_{qm} = 0 = i^{mc}_{qs} + i\,'^{mc}_{qr}$$

$$\psi^{mc}_{ds} = x_s i^{mc}_{ds} + x_M i\,'^{mc}_{dr}$$

$$\psi^{mc}_{qs} = x_s i^{mc}_{qs} + x_M i\,'^{mc}_{qr} = x_{ls} i^{mc}_{qs}$$

$$\psi\,'^{mc}_{qr} = x_M i^{mc}_{qs} + x\,'_r i\,'^{mc}_{qr} = x\,'_{lr} i\,'^{mc}_{qr}$$

$$\psi\,'^{mc}_{dr} = x\,'_r i\,'^{mc}_{dr} + x_M i^{mc}_{ds}$$

$$\frac{d\Omega}{dt} = \frac{1}{2H} \cdot \left(\frac{3P}{4} x_M \cdot i^{mc}_{dm} \cdot i^{mc}_{qs} - T_L \right) \tag{177-185}$$

A block diagram for the mathematical model of the induction machine in this reference frame is presented in Fig. 2.15. As inputs are chosen the stator current space vector, the d-axis component of the stator voltage space vector, the synchronous angular velocity and the load torque. The output is the machine (rotor) angular velocity. The similarity to the synchronous reference frame linked to the air-gap flux space vector can be observed, as the magnetising current is directly proportional with the air-gap flux, through the magnetising inductance.

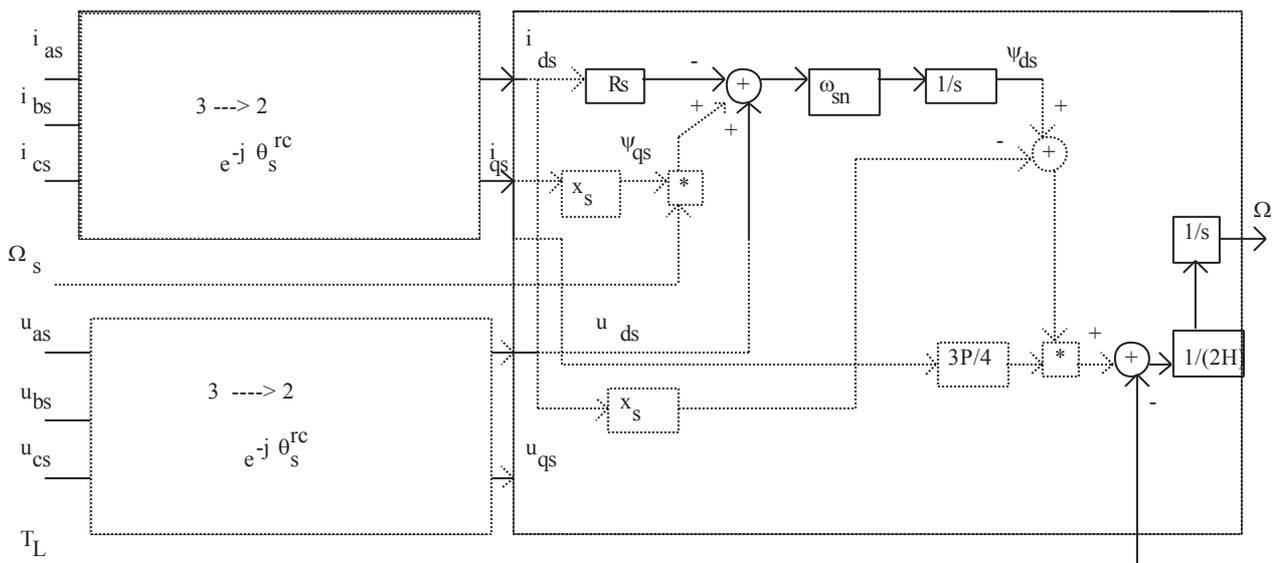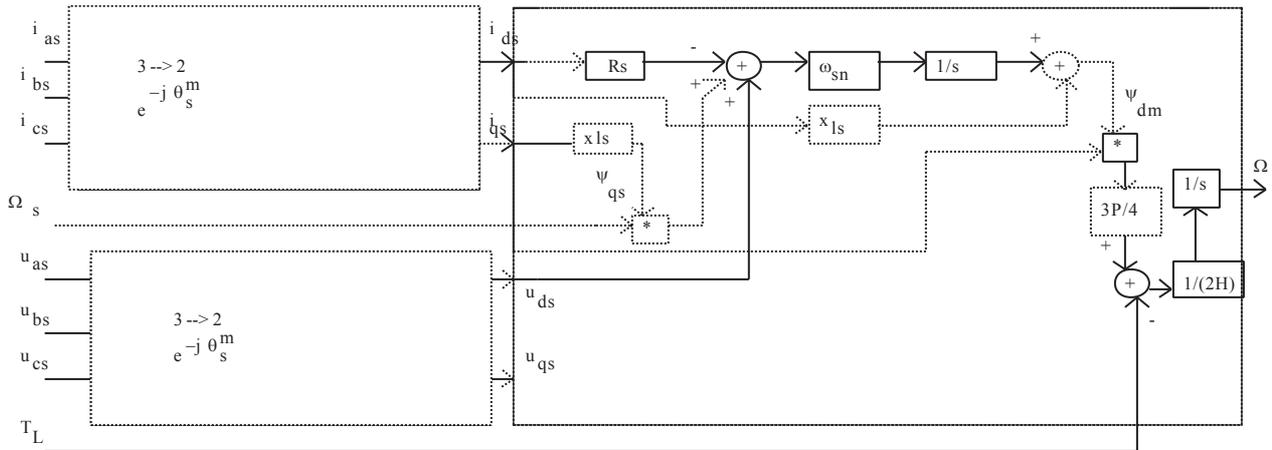

Fig. 2.15. Block diagram of the induction machine in synchronous reference frame
(linked to magnetising current space vector)

2.8. D-Q Axes models of the three-phase induction machine

The three-phase induction machine can be modelled by using different state-space variables and keeping as inputs the stator voltages and the load torque, and as outputs the electromagnetic torque and the rotor angular velocity. The possible set of currents and flux linkages per second space vectors are defined as follows (Nowotny and Lipo - 1996):

$$\underline{i}_s = \underline{i}_{ds} + j \cdot i_{qs}$$

$$\underline{i}_r = i'_{dr} + j \cdot i'_{qr}$$

$$\underline{i}_m = \underline{i}_{dm} + j \cdot i_{qm}$$

$$\underline{\lambda}_s = \lambda_{ds} + j \cdot \lambda_{qs} \tag{186-191}$$

$$\underline{\lambda}'_r = \lambda'_{dr} + j \cdot \lambda'_{qr}$$

$$\underline{\lambda}_m = \lambda_{dm} + j \cdot \lambda_{qm}$$

$$[x] = \begin{bmatrix} \underline{i}_s & \underline{i}_r & \underline{i}_m & \underline{\lambda}_s & \underline{\lambda}'_r & \underline{\lambda}_m \end{bmatrix}^{\mathrm{T}} \tag{192}$$

The $d\text{-}q$ axes are orthogonal and fixed to the stator, d axis coincides to the magnetic axis of the as winding.

As there are four voltage equations, it is necessary to consider two of the space vectors as state-variables in order to obtain a solution for the equations system. Let the selected pair of state-space variables be denoted as $\underline{x}_1, \underline{x}_2$. The set of six state-space variables will be expressed in terms of the two selected state-space variables:

$$[x] = \begin{bmatrix} \underline{i}_s & \underline{i}'_r & \underline{i}_m & \underline{\lambda}_s & \underline{\lambda}'_r & \underline{\lambda}_m \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_1 & \underline{x}_2 \end{bmatrix}^{\mathrm{T}} \tag{193}$$

Only fourteen out of the fifteen state-space possibilities represent valid mathematical models for the three-phase induction machine (the pair of state-space variables that comprises airgap flux linkage per second space vector and the magnetising current space vector cannot be selected as these vectors have the same direction). There are three types of models:
1. current state-space variables models;
2. flux linkages state-space variables models;
3. mixed currents-flux linkages state-space variable models.

In conjunction with the mechanical equation:

$$p\omega_r = \frac{P}{2J} \cdot (T_e - T_L) \tag{194}$$

we obtain a complete version of the three-phase induction machine model, viewed as the key for a motion control system.

The starting point for the state-variable models is given by the voltage equations system written in *stationary reference frame*:

$$u = L \cdot px + R \cdot x \tag{195}$$

where: x is the selected set of state-variables and represents also the output of the model, u is the input vector (stator voltages), L is the coefficients matrix (it can be formed by reactance values, or non-dimensional elements) for multiplying the time derivative of the state-variables, R is the coefficients matrix (it can be formed by resistances and reactances values or non-dimensional elements) for multiplying the state-variables and p stands for the differential operator (d/dt).

It results the general form of the state-variables system:

$$px = A \cdot x + B \cdot u \qquad (196)$$

where:

$$A = -L^{-1} \cdot R$$

$$B = L^{-1}$$

All the following mathematical models permit a discretisation for the implementation of controllers in the drive systems with three-phase induction machines. Classically, the current state-space and the flux state-space models are the chosen options for the manufacturers of drive systems.

The original vector orientation scheme was based on the alignment of the synchronous reference frame to the rotor flux linkage space vector. Then the vector control strategy was extended by considering as well the stator and air-gap flux space vectors, as alignment of the reference frame. However, for a complete analysis of the vector control strategies, each of the six space vectors that can be selected as state-space variables represents a possible basis for a reference frame. If the magnetising current space vector selection as alignment gives similar results to the air-gap flux space vector case, the stator and rotor current space vectors are still to be further analysed as new options for vector control.

An easy to follow steps algorithm for implementing vector oriented control systems is obtained as follows:

1. A complete mathematical model of the three-phase induction machine is developed in stationary reference frame, according to the chosen set of state-space variables;
2. The rotor based variables are completely expressed in the new state-variable system;
3. The rotor angular velocity ω_r term is substituted with $(\omega_r - \omega_k)$ where ω_k is the angular velocity of the synchronous frame;
4. The synchronous reference frame is selected linked to one of the space vectors, which means that the q-axis component of the reference space vector is null;
5. The torque equation is computed according to the selected flux or current space vector in the synchronous reference frame.

The block diagram of modelling an induction motor for vector control purpose is detailed in Fig. 2.16.
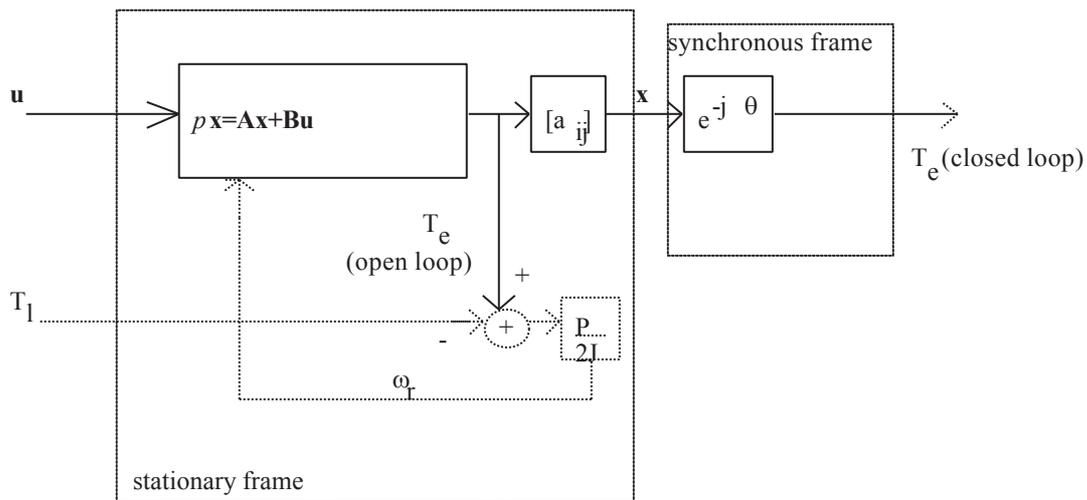
Fig. 2.16. Block diagram form of the induction machine mathematical model

Fig. 2.17 shows the general two-axis equivalent circuit for the induction machine, in an arbitrary reference frame. This equivalent circuit represents the starting point for determining the matrix equations, which will be derived further.
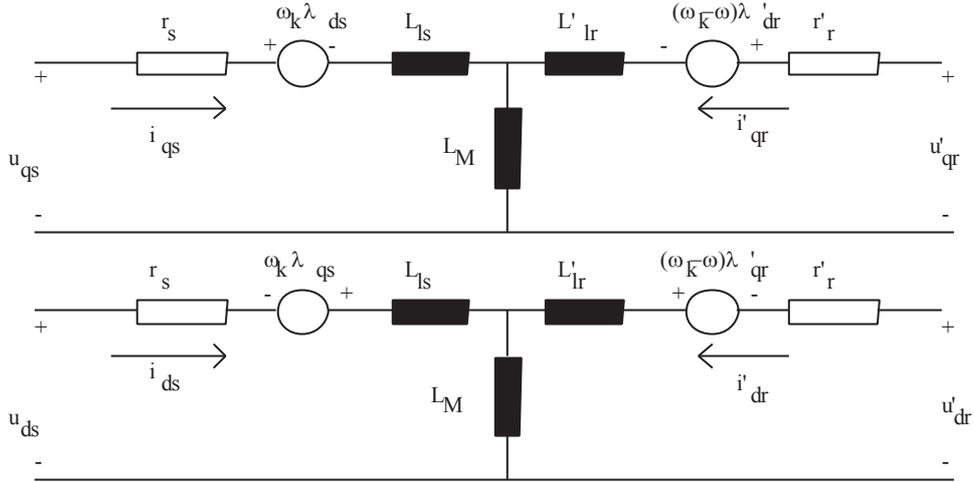
Fig. 2.17. Arbitrary reference-frame equivalent circuits for a three-phase, symmetrical induction machine

Note: For compactness, the elements from Fig. 2.17. have to be transformed in terms of reactances, while the flux linkage has to be expressed in flux linkage units per second, or volts.

2.8.1. Models with currents space vectors state-space variables

I) As complex state variables, the currents space vectors $x = \left[i_{qs}, i_{ds}, i'_{qr}, i'_{dr} \right]^{T}$ are usually assumed. The stator current space vector is considered generally as the right choice, because it corresponds to directly measurable quantities. This model is readily available from voltages and flux linkages per second equations, and it can be expressed in a matrix form as follows:

$$
\frac{p}{\omega_b} \cdot \begin{bmatrix} i_{qs} \\ i_{ds} \\ i'_{qr} \\ i'_{dr} \end{bmatrix} = \begin{bmatrix}
-\dfrac{r_s x'_r}{D} & -\dfrac{x_m^2}{D} \cdot \dfrac{\omega_r}{\omega_b} & \dfrac{x_m r'_r}{D} & -\dfrac{x_m x'_r}{D} \cdot \dfrac{\omega_r}{\omega_b} \\[2mm]
\dfrac{x_m^2}{D} \cdot \dfrac{\omega_r}{\omega_b} & -\dfrac{r_s x'_r}{D} & \dfrac{x_m x'_r}{D} \cdot \dfrac{\omega_r}{\omega_b} & \dfrac{x_m r'_r}{D} \\[2mm]
\dfrac{x_m r_s}{D} & \dfrac{x_s x_m}{D} \cdot \dfrac{\omega_r}{\omega_b} & -\dfrac{x_s r'_r}{D} & \dfrac{x_s x'_r}{D} \cdot \dfrac{\omega_r}{\omega_b} \\[2mm]
-\dfrac{x_s x_m}{D} \cdot \dfrac{\omega_r}{\omega_b} & \dfrac{x_m r_s}{D} & -\dfrac{x_s x'_r}{D} \cdot \dfrac{\omega_r}{\omega_b} & -\dfrac{x_s r'_r}{D}
\end{bmatrix} \cdot \begin{bmatrix} i_{qs} \\ i_{ds} \\ i'_{qr} \\ i'_{dr} \end{bmatrix} + \begin{bmatrix}
\dfrac{x'_r}{D} & 0 & \dfrac{-x_m}{D} & 0 \\[2mm]
0 & \dfrac{x'_r}{D} & 0 & -\dfrac{x_m}{D} \\[2mm]
-\dfrac{x_m}{D} & 0 & \dfrac{x_s}{D} & 0 \\[2mm]
0 & -\dfrac{x_m}{D} & 0 & \dfrac{x_s}{D}
\end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}
$$

(197)

where: $D = x_s x'_r - x_m^2$

If the stator, rotor and total leakage factor definitions are used:

$$x_s = x_{ls} + x_m = (1 + \sigma_s) x_m$$

$$x'_r = x'_{lr} + x_m = (1 + \sigma_r) x_m$$

$$\sigma = 1 - \frac{1}{(1 + \sigma_s)(1 + \sigma_r)}$$

we obtain:

$$D = x_m^2 \cdot (\sigma_s \sigma_r + \sigma_s + \sigma_r) = x_m^2 \cdot \frac{\sigma}{1 - \sigma}$$

The electromagnetic torque can be computed as:

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b} \cdot \left(i_{qs} i'_{dr} - i_{ds} i'_{qr} \right) \tag{198}$$

II) The stator and the magnetising space vector currents as state-space variables $x = \left[i_{qs}, i_{ds}, i_{qm}, i_{dm} \right]^T$ represent another mathematical *d-q* axis model based on currents space vectors. By selecting the magnetising current space vector among the state-space variables, it is possible to include the saturation effect in modelling the induction motor. Also, the stator current space vector is a measurable quantity, and determines a precise and accurate option for implementing controllers. The state matrix elements are all non-zero, which implies a computational effort similar to the previously analysed model.

$$
\frac{p}{\omega_b} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qm} \\ i_{dm} \end{bmatrix} = \frac{1}{D} \cdot
\begin{bmatrix}
-r_s x'_r - r'_r x_m & x_m x'_{lr} \cdot \dfrac{\omega_r}{\omega_b} & x_m r'_r & -x_m x'_r \cdot \dfrac{\omega_r}{\omega_b} \\
-x_m x'_{lr} \cdot \dfrac{\omega_r}{\omega_b} & -r_s x'_r - r'_r x_m & x_m x'_r \cdot \dfrac{\omega_r}{\omega_b} & x_m r'_r \\
r'_r x_{ls} - r_s x'_{lr} & -x_{ls} x'_{lr} \cdot \dfrac{\omega_r}{\omega_b} & -x_{ls} r'_r & x_{ls} x'_r \cdot \dfrac{\omega_r}{\omega_b} \\
x_{ls} x'_{lr} \cdot \dfrac{\omega_r}{\omega_b} & r'_r x_{ls} - r_s x'_{lr} & -x_{ls} x'_r \cdot \dfrac{\omega_r}{\omega_b} & -x_{ls} r'_r
\end{bmatrix}
\cdot \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qm} \\ i_{dm} \end{bmatrix} +
$$

$$
+ \frac{1}{D} \cdot \begin{bmatrix} x'_r & 0 & -x_m & 0 \\ 0 & x'_r & 0 & -x_m \\ x'_{lr} & 0 & x_{ls} & 0 \\ 0 & x'_{lr} & 0 & x_{ls} \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix} \tag{199}
$$

The instantaneous electromagnetic torque may be expressed in terms of the state-space variables:

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b} \cdot \left(i_{qs} i_{dm} - i_{ds} i_{qm} \right) \tag{200}$$

III) The third possible combination of current space vectors as state-space variables is obtained by selecting magnetising and rotor currents $x = \left[i'_{qr}, i'_{dr}, i_{qm}, i_{dm} \right]^T$. When compared to the other currents space vector models, the similar computational burden for obtaining the outputs of the system is obvious. The main difference between them is the presence of the global parameters (self-reactances) in the stator and rotor current state-space variables model, while in the other two models an accurate determination of the leakage reactances or leakage factors is necessary. A complete description of the state-variables in matrix notation is given below:

$$\frac{p}{\omega_b}\begin{bmatrix} i'_{qr} \\ i'_{dr} \\ i_{qm} \\ i_{dm} \end{bmatrix} = \frac{1}{D} \cdot \begin{bmatrix} -r_s x_m - r'_r x_s & x_s x'_{lr}\dfrac{\omega_r}{\omega_b} & r_s x_m & x_s x_m \dfrac{\omega_r}{\omega_b} \\[2mm] -x_s x'_{lr}\dfrac{\omega_r}{\omega_b} & -r_s x_m - r'_r x_s & x_s x_m \dfrac{\omega_r}{\omega_b} & r_s x_m \\[2mm] r_s x'_{lr} - r'_r x_{ls} & x_{ls} x'_{lr}\dfrac{\omega_r}{\omega_b} & -r_s x'_{lr} & x_{ls} x_m \dfrac{\omega_r}{\omega_b} \\[2mm] -x_{ls} x'_{lr}\dfrac{\omega_r}{\omega_b} & r_s x'_{lr} - r'_r x_{ls} & -x_{ls} x_m \dfrac{\omega_r}{\omega_b} & -r_s x'_{lr} \end{bmatrix} \begin{bmatrix} i'_{qr} \\ i'_{dr} \\ i_{qm} \\ i_{dm} \end{bmatrix} +$$

$$+\frac{1}{D}\begin{bmatrix} -x_m & 0 & x_s & 0 \\ 0 & -x_m & 0 & x_s \\ x'_{lr} & 0 & x_{ls} & 0 \\ 0 & x'_{lr} & 0 & x_{ls} \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}$$

(201)

The instantaneous electromagnetic torque is expressed as follows:

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b} \cdot \left(i'_{dr}\, i_{qm} - i'_{qr}\, i_{dm} \right)$$

(202)

2.8.2. Models with flux linkages as state-space varaibles

I) When flux linkages per second are selected as state-space variables, the models are less computationally demanding than in the currents state-space variables version. As each flux contains information about two currents space vectors components, the state matrix contains zero elements. One option is to select the stator and rotor flux linkages space vectors $x = \left[\lambda_{qs}, \lambda_{ds}, \lambda'_{qr}, \lambda'_{dr} \right]^T$ for describing the mathematical model of the machine. For this system the matrix equations are as follows:

$$\frac{p}{\omega_b} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} = \begin{bmatrix} \dfrac{-r_s x'_r}{D} & 0 & \dfrac{r_s x_m}{D} & 0 \\[3mm] 0 & \dfrac{-r_s x'_r}{D} & 0 & \dfrac{r_s x_m}{D} \\[3mm] \dfrac{r'_r x_m}{D} & 0 & \dfrac{-r'_r x_s}{D} & \dfrac{\omega_r}{\omega_b} \\[3mm] 0 & \dfrac{r'_r x_m}{D} & \dfrac{\omega_r}{\omega_b} & \dfrac{-r'_r x_s}{D} \end{bmatrix} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} + \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}$$

(203)

The electromagnetic torque is determined with the equivalent relation:

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b \cdot D} \cdot \left(\lambda_{qs}\lambda'_{dr} - \lambda_{ds}\lambda'_{qr} \right)$$

(204)

By comparison with the current models, it can be observed that the computational burden is essentially lower. Due to this important feature, this model is the most suitable for discretisation in motion control strategies.

II) An alternative to model the three-phase induction machine with flux space vectors as state-variable system, is the selection of air-gap flux space vector among the set of independent variables. This choice presents the advantage of an easier saturation effect modelling, but the disadvantage of an increased computational burden. The air-gap flux is a measurable quantity, and this advantage imposes it in many practical solutions for vector control schemes.

A first approach is given by the stator and air-gap flux space vectors selected as state-space variables $\boldsymbol{x} = \left[\lambda_{qs}, \lambda_{ds}, \lambda_{qm}, \lambda_{dm} \right]^{\mathrm{T}}$.

$$
\frac{p}{\omega_b}
\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{qm} \\ \lambda_{dm} \end{bmatrix}
=
\begin{bmatrix}
-\dfrac{r_s}{x_{ls}} & 0 & \dfrac{r_s}{x_{ls}} & 0 \\[2mm]
0 & -\dfrac{r_s}{x_{ls}} & 0 & \dfrac{r_s}{x_{ls}} \\[2mm]
-\dfrac{r_s x'_{lr} x_m}{x_{ls} D} + \dfrac{r'_r x_m}{D} & -\dfrac{x_m x'_{lr}}{D}\cdot\dfrac{\omega_r}{\omega_b} & \dfrac{r_s x_m x'_{lr}}{x_{ls} D} - \dfrac{r'_r x_s}{D} & \dfrac{x_{ls} x_m}{D}\cdot\dfrac{\omega_r}{\omega_b}\cdot\left(1+\dfrac{x'_{lr} x_s}{x_{ls} x_m}\right) \\[2mm]
\dfrac{x_m x'_{lr}}{D}\cdot\dfrac{\omega_r}{\omega_b} & -\dfrac{r_s x'_{lr} x_m}{x_{ls} D} + \dfrac{r'_r x_m}{D} & -\dfrac{x_{ls} x_m}{D}\cdot\dfrac{\omega_r}{\omega_b}\cdot\left(1+\dfrac{x'_{lr} x_s}{x_{ls} x_m}\right) & \dfrac{r_s x'_{lr} x_m}{x_{ls} D} - \dfrac{r'_r x_s}{D}
\end{bmatrix}
\cdot
\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{qm} \\ \lambda_{dm} \end{bmatrix}
+
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\[2mm]
0 & 1 & 0 & 0 \\[2mm]
\dfrac{x'_{lr} x_m}{D} & 0 & \dfrac{x_{ls} x_m}{D} & 0 \\[2mm]
0 & \dfrac{x'_{lr} x_m}{D} & 0 & \dfrac{x_{ls} x_m}{D}
\end{bmatrix}
\cdot
\begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}
$$

(205)

The electromagnetic torque expression depends on stator leakage factor:

$$
T_e = \frac{3P}{4}\cdot\frac{1}{x_{ls}\omega_b}\cdot\left[\lambda_{qs}\lambda_{dm} - \lambda_{ds}\lambda_{qm} \right]
\tag{206}
$$

III) The third option of selecting flux space vectors as state-space variables is similar to the previous model. It comprises the air-gap (magnetising) flux and rotor flux space vectors $\boldsymbol{x} = \left[\lambda_{qm}, \lambda_{dm}, \lambda'_{qr}, \lambda'_{dr} \right]^{\mathrm{T}}$ in the set of independent variables. The same computational effort is required, and the electromagnetic torque is determined in a suitable form for vector control if the rotor leakage factor is known:

$$
\frac{p}{\omega_b}
\begin{bmatrix} \lambda_{qm} \\ \lambda_{dm} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix}
=
\begin{bmatrix}
-\dfrac{r_s x'_r}{D} + \dfrac{r'_r x_m x_{ls}}{x'_{lr} D} & 0 & \dfrac{r_s x_m}{D} - \dfrac{r'_r x_m x_{ls}}{x'_{lr} D} & \dfrac{x_m x_{ls}}{D}\cdot\dfrac{\omega_r}{\omega_b} \\[2mm]
0 & -\dfrac{r_s x'_r}{D} + \dfrac{r'_r x_m x_{ls}}{x'_{lr} D} & -\dfrac{x_m x_{ls}}{D}\cdot\dfrac{\omega_r}{\omega_b} & \dfrac{r_s x_m}{D} - \dfrac{r'_r x_m x_{ls}}{x'_{lr} D} \\[2mm]
\dfrac{r'_r}{x'_{lr}} & 0 & -\dfrac{r'_r}{x'_{lr}} & \dfrac{\omega_r}{\omega_b} \\[2mm]
0 & \dfrac{r'_r}{x'_{lr}} & -\dfrac{\omega_r}{\omega_b} & -\dfrac{r'_r}{x'_{lr}}
\end{bmatrix}
\cdot
\begin{bmatrix} \lambda_{qm} \\ \lambda_{dm} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix}
+
$$

$$
+
\begin{bmatrix}
\dfrac{x_m x'_{lr}}{D} & 0 & \dfrac{x_m x_{ls}}{D} & 0 \\[2mm]
0 & \dfrac{x_m x'_{lr}}{D} & 0 & \dfrac{x_m x_{ls}}{D} \\[2mm]
0 & 0 & 1 & 0 \\[2mm]
0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}
$$

(207)

The resulting relation for computing the instantaneous electromagnetic torque is as follows:

$$T_e = \frac{3P}{4} \cdot \frac{1}{x'_{lr}\,\omega_b} \cdot \left(\lambda_{qm}\lambda'_{dr} - \lambda_{dm}\lambda'_{qr} \right) \qquad (208)$$

2.8.3. Models with mixed currents – flux space vectors state-space variables

I) For obtaining an acceptable computational effort as well as measurable output quantities, the general accepted solutions are the mixed currents-flux state-space variable models. If the stator variables are chosen for modelling the three-phase induction machine system, then a mixed flux linkages-currents state-space variables model $x = \left[\lambda_{qs}, \lambda_{ds}, i_{qs}, i_{ds} \right]^T$ is developed. This mathematical model is selected when stator flux oriented strategy is implemented. The matrix equations and the electromagnetic torque relation are:

$$\frac{p}{\omega_b}\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ i_{qs} \\ i_{ds} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -r_s & 0 \\ 0 & 0 & 0 & -r_s \\ \dfrac{r'_r}{D} & -\dfrac{x'_r}{D}\cdot\dfrac{\omega_r}{\omega_b} & -\dfrac{r_s x'_r + r'_r x_s}{D} & \dfrac{\omega_r}{\omega_b} \\ \dfrac{x'_r}{D}\cdot\dfrac{\omega_r}{\omega_b} & \dfrac{r'_r}{D} & -\dfrac{\omega_r}{\omega_b} & -\dfrac{r_s x'_r + r'_r x_s}{D} \end{bmatrix} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ i_{qs} \\ i_{ds} \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{x'_r}{D} & 0 & -1 & 0 \\ 0 & \dfrac{x'_r}{D} & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix} \qquad (209)$$

The instantaneous electromagnetic torque relation depends only on the magnitude of the output vector components.

$$T_e = \frac{3P}{4} \cdot \frac{1}{\omega_b} \cdot \left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds} \right) \qquad (210)$$

II) Another important mixed flux linkages-currents state-space variables model $x = \left[\lambda'_{qr}, \lambda'_{dr}, i'_{qr}, i'_{dr} \right]^T$ is that expressed in rotor quantities. This model represents the optimum solution for rotor flux oriented control strategy in a drive system with an induction machine. However, as it is impossible to measure the rotor currents if the machine is equipped with cage rotor, there are limitations in using this model for vector control strategies. The state-space variables system is detailed below:

$$\frac{p}{\omega_b}\begin{bmatrix} \lambda'_{qr} \\ \lambda'_{dr} \\ i'_{qr} \\ i'_{dr} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\omega_r}{\omega_b} & -r'_r & 0 \\ -\dfrac{\omega_r}{\omega_b} & 0 & 0 & -r'_r \\ \dfrac{r_s}{D} & \dfrac{x_s}{D}\cdot\dfrac{\omega_r}{\omega_b} & -\dfrac{r_s x'_r + r'_r x_s}{D} & 0 \\ -\dfrac{x_s}{D}\cdot\dfrac{\omega_r}{\omega_b} & \dfrac{r_s}{D} & 0 & -\dfrac{r_s x'_r + r'_r x_s}{D} \end{bmatrix} \cdot \begin{bmatrix} \lambda'_{qr} \\ \lambda'_{dr} \\ i'_{qr} \\ i'_{dr} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{x_m}{D} & 0 & 0 & 0 \\ 0 & -\dfrac{x_m}{D} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}$$

$$(211)$$

The electromagnetic torque expression depends only on the magnitude of the output vector components:

$$T_e = \frac{3P}{4} \cdot \frac{1}{\omega_b} \left(\lambda'_{qr} \, i'_{dr} - \lambda'_{dr} \, i'_{qr} \right) \tag{212}$$

III) The most used alternative model for rotor flux oriented control strategies is that which comprises the stator current and rotor flux linkage $x = \left[\lambda'_{qr}, \lambda'_{dr}, i_{qs}, i_{ds} \right]^T$ as state-space variables. It contains the advantages of measurable output quantities (stator currents) and acceptable computational burden. The matrix equation and the electromagnetic torque relation are presented below:

$$\frac{p}{\omega_b} \begin{bmatrix} i_{qs} \\ i_{ds} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_s x'^2_r + r'_r x^2_m}{x'_r D} & 0 & \dfrac{r'_r x_m}{x'_r D} & -\dfrac{x_m}{D} \cdot \dfrac{\omega_r}{\omega_b} \\ 0 & -\dfrac{r_s x'^2_r + r'_r x^2_m}{x'_r D} & \dfrac{x_m}{D} \cdot \dfrac{\omega_r}{\omega_b} & \dfrac{r'_r x_m}{x'_r D} \\ \dfrac{r'_r x_m}{x'_r} & 0 & -\dfrac{r'_r}{x'_r} & \dfrac{\omega_r}{\omega_b} \\ 0 & \dfrac{r'_r x_m}{x'_r} & -\dfrac{\omega_r}{\omega_b} & -\dfrac{r'_r}{x'_r} \end{bmatrix} \cdot \begin{bmatrix} i_{qs} \\ i_{ds} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} +$$

$$+ \begin{bmatrix} \dfrac{x'_r}{D} & 0 & -\dfrac{x_m}{D} & 0 \\ 0 & \dfrac{x'_r}{D} & 0 & -\dfrac{x_m}{D} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix} \tag{213}$$

and

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{x'_r \omega_b} \cdot \left(i_{qs} \lambda'_{dr} - i_{ds} \lambda'_{qr} \right) \tag{214}$$

IV) A theoretical mathematical model is that with mixed rotor currents space vector and stator flux linkages space vector as state-space variables $x = \left[\lambda_{qs}, \lambda_{ds}, i'_{qr}, i'_{dr} \right]^T$. It should be noted that the advantage of this model is that only the stator winding parameter is necessary and as such the influence of rotor parameters is minimised. This model can be used for an unconventional stator flux oriented control with rotor current components producing the torque and the flux. The expressions for implementing this model are:

$$\frac{p}{\omega_b}\begin{bmatrix} i'_{qr} \\ i'_{dr} \\ \lambda_{qs} \\ \lambda_{ds} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_s x_m^2 + r'_r x_s^2}{x_s D} & \dfrac{\omega_r}{\omega_b} & \dfrac{r_s x_m}{x_s D} & \dfrac{x_m}{D}\cdot\dfrac{\omega_r}{\omega_b} \\[2ex] -\dfrac{\omega_r}{\omega_b} & -\dfrac{r_s x_m^2 + r'_r x_s^2}{x_s D} & -\dfrac{x_m}{D} & \dfrac{r_s x_m}{x_s D} \\[2ex] \dfrac{r_s x_m}{x_s} & 0 & -\dfrac{r_s}{x_s} & 0 \\[2ex] 0 & \dfrac{r_s x_m}{x_s} & 0 & -\dfrac{r_s}{x_s} \end{bmatrix}\cdot\begin{bmatrix} i'_{qr} \\ i'_{dr} \\ \lambda_{qs} \\ \lambda_{ds} \end{bmatrix} + \begin{bmatrix} \dfrac{x_m}{D} & 0 & \dfrac{x_s}{D} & 0 \\[2ex] 0 & \dfrac{x_m}{D} & 0 & \dfrac{x_s}{D} \\[2ex] 1 & 0 & 0 & 0 \\[2ex] 0 & 1 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix} \quad (215)$$

and

$$T_e = \frac{3P}{4}\cdot\frac{x_m}{x_s \omega_b}\cdot\left(\lambda_{qs} i'_{dr} - \lambda_{ds} i'_{qr}\right) \tag{216}$$

V) The mixed stator current space vector and air-gap flux space vector $x = \left[\lambda_{qm}, \lambda_{dm}, i_{qs}, i_{ds}\right]^T$ as state-space variables belongs to one of the most complex types of models. It preserves information regarding both stator and rotor parameters. This mathematical model is the suitable choice for the air-gap flux orientation control strategy. The greatest advantage of this model is that by using Hall sensors or tapped stator windings, all the output vector components are measurable. Due to its versatility, this model is widely used in controllers implementation, especially for medium speed applications. It permits also the simulation or modelling of the saturation effect. By comparison to the previous mixed models, the state-space matrix contains only non-zero elements, which leads to greater computational effort.

$$\frac{p}{\omega_b}\begin{bmatrix} i_{qs} \\ i_{ds} \\ \lambda_{qm} \\ \lambda_{dm} \end{bmatrix} = \frac{1}{D}\cdot\begin{bmatrix} -r_s x'_r - r'_r x_m & x_m x'_{lr}(\omega_r/\omega_b) & r'_r & -x'_r(\omega_r/\omega_b) \\ x_m x'_{lr}(\omega_r/\omega_b) & -r_s x'_r - r'_r x_m & x'_r(\omega_r/\omega_b) & r'_r \\ x_m(r'_r x_m - r_s x'_{lr}) & -x_{ls} x_m x'_{lr}(\omega_r/\omega_b) & -x_{ls} r'_r & x_{ls} x'_r(\omega_r/\omega_b) \\ -x_{ls} x_m x'_{lr}(\omega_r/\omega_b) & x_m(r'_r x_m - r_s x'_{lr}) & -x_{ls} x'_r(\omega_r/\omega_b) & -x_{ls} r'_r \end{bmatrix}\cdot\begin{bmatrix} i_{qs} \\ i_{ds} \\ \lambda_{qm} \\ \lambda_{dm} \end{bmatrix} +$$

$$+\frac{1}{D}\cdot\begin{bmatrix} x'_r & 0 & -x_m & 0 \\ 0 & x'_r & 0 & -x_m \\ x'_{lr} x_m & 0 & x_{ls} x_m & 0 \\ 0 & x'_{lr} x_m & 0 & x_{ls} x_m \end{bmatrix}\cdot\begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}$$

$$\tag{217}$$

The electromagnetic torque is computed as:

$$T_e = \frac{3P}{4}\cdot\frac{1}{\omega_b}\cdot\left(i_{qs}\lambda_{dm} - i_{ds}\lambda_{qm}\right) \tag{218}$$

VI) Another theoretical model, similar in form to the precedent one, is the mixed rotor current space vector and air-gap flux space vector as state-space variables $x = \left[\lambda_{qm}, \lambda_{dm}, i'_{qr}, i'_{dr}\right]^T$ Its main shortcoming is the presence of the unmeasurable rotor currents among the state-space variables. The matrix equation of the system is given below:

$$\frac{p}{\omega_{\mathrm{b}}}\begin{bmatrix} i'_{\mathrm{qr}} \\ i'_{\mathrm{dr}} \\ \lambda_{\mathrm{qm}} \\ \lambda_{\mathrm{dm}} \end{bmatrix} = \frac{1}{D} \cdot \begin{bmatrix} -r_{\mathrm{s}}x_{\mathrm{m}} - r'_{\mathrm{r}}x_{\mathrm{s}} & x_{\mathrm{s}}x'_{\mathrm{lr}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) & r_{\mathrm{s}} & x_{\mathrm{s}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) \\ -x_{\mathrm{s}}x'_{\mathrm{lr}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) & -r_{\mathrm{s}}x_{\mathrm{m}} - r'_{\mathrm{r}}x_{\mathrm{s}} & -x_{\mathrm{s}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) & r_{\mathrm{s}} \\ r_{\mathrm{s}}x_{\mathrm{m}}x'_{\mathrm{lr}} - r'_{\mathrm{r}}x_{\mathrm{m}}x_{\mathrm{ls}} & x_{\mathrm{m}}x_{\mathrm{ls}}x'_{\mathrm{lr}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) & -r_{\mathrm{s}}x'_{\mathrm{lr}} & x_{\mathrm{m}}x_{\mathrm{ls}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) \\ -x_{\mathrm{m}}x_{\mathrm{ls}}x'_{\mathrm{lr}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) & r_{\mathrm{s}}x_{\mathrm{m}}x'_{\mathrm{lr}} - r'_{\mathrm{r}}x_{\mathrm{m}}x_{\mathrm{ls}} & -x_{\mathrm{m}}x_{\mathrm{ls}}(\omega_{\mathrm{r}}/\omega_{\mathrm{b}}) & -r_{\mathrm{s}}x'_{\mathrm{lr}} \end{bmatrix} \cdot \begin{bmatrix} i'_{\mathrm{qr}} \\ i'_{\mathrm{dr}} \\ \lambda_{\mathrm{qm}} \\ \lambda_{\mathrm{dm}} \end{bmatrix} +$$

$$+ \frac{1}{D} \cdot \begin{bmatrix} -x_{\mathrm{m}} & 0 & x_{\mathrm{s}} & 0 \\ 0 & -x_{\mathrm{m}} & 0 & x_{\mathrm{s}} \\ x_{\mathrm{m}}x'_{\mathrm{lr}} & 0 & x_{\mathrm{m}}x_{\mathrm{ls}} & 0 \\ 0 & x_{\mathrm{m}}x'_{\mathrm{lr}} & 0 & x_{\mathrm{m}}x_{\mathrm{ls}} \end{bmatrix} \cdot \begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

$$(219)$$

The electromagnetic torque relation becomes:

$$T_{\mathrm{e}} = \frac{3P}{4} \cdot \frac{1}{\omega_{\mathrm{b}}} \cdot \left(i_{\mathrm{dr}}\lambda_{\mathrm{qm}} - i_{\mathrm{qr}}\lambda_{\mathrm{dm}} \right) \qquad (220)$$

VII) If the magnetising current space vector is selected as state-space variable together with one of the flux linkages space vectors (i.e. stator flux linkages) $x = \left[i_{\mathrm{qm}}, i_{\mathrm{dm}}, \lambda_{\mathrm{qs}}, \lambda_{\mathrm{ds}} \right]^{\mathrm{T}}$, the computation of the state matrix elements gives several null results. This choice for a set of state-space variables presents only theoretical importance, as the output vector components are unmeasurable. Also the computational demand of the model does not make it a practical option for vector control implementation.

$$\frac{p}{\omega_{\mathrm{b}}}\begin{bmatrix} i_{\mathrm{qm}} \\ i_{\mathrm{dm}} \\ \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \end{bmatrix} = \begin{bmatrix} \dfrac{r_{\mathrm{s}}x'_{\mathrm{lr}}x_{\mathrm{m}} - r'_{\mathrm{r}}x_{\mathrm{s}}x_{\mathrm{ls}}}{x_{\mathrm{ls}}D} & \dfrac{\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & \dfrac{r'_{\mathrm{r}}x_{\mathrm{ls}} - r_{\mathrm{s}}x'_{\mathrm{lr}}}{x_{\mathrm{ls}}D} & -\dfrac{x'_{\mathrm{lr}}}{D}\cdot\dfrac{\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} \\[3mm] -\dfrac{\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & \dfrac{r_{\mathrm{s}}x_{\mathrm{m}}x'_{\mathrm{lr}} - r'_{\mathrm{r}}x_{\mathrm{s}}x_{\mathrm{ls}}}{x_{\mathrm{ls}}D} & \dfrac{x'_{\mathrm{lr}}}{D}\cdot\dfrac{\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & \dfrac{r'_{\mathrm{r}}x_{\mathrm{ls}} - r_{\mathrm{s}}x'_{\mathrm{lr}}}{x_{\mathrm{ls}}D} \\[3mm] \dfrac{r_{\mathrm{s}}x_{\mathrm{m}}}{x_{\mathrm{ls}}} & 0 & -\dfrac{r_{\mathrm{s}}}{x_{\mathrm{ls}}} & 0 \\[3mm] 0 & \dfrac{r_{\mathrm{s}}x_{\mathrm{m}}}{x_{\mathrm{ls}}} & 0 & -\dfrac{r_{\mathrm{s}}}{x_{\mathrm{ls}}} \end{bmatrix} \cdot \begin{bmatrix} i_{\mathrm{qm}} \\ i_{\mathrm{dm}} \\ \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \end{bmatrix} +$$

$$+ \begin{bmatrix} \dfrac{x'_{\mathrm{lr}}}{D} & 0 & \dfrac{x_{\mathrm{ls}}}{D} & 0 \\[3mm] 0 & \dfrac{x'_{\mathrm{lr}}}{D} & 0 & \dfrac{x_{\mathrm{ls}}}{D} \\[3mm] 1 & 0 & 0 & 0 \\[2mm] 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

$$(221)$$

The electromagnetic torque relation:

$$T_{\mathrm{e}} = \frac{3P}{4} \cdot \frac{x_{\mathrm{m}}}{\omega_{\mathrm{b}}x_{\mathrm{ls}}} \cdot \left(i_{\mathrm{dm}}\lambda_{\mathrm{qs}} - i_{\mathrm{qm}}\lambda_{\mathrm{ds}} \right) \qquad (222)$$

VIII) One other model analysed for theoretical reasons, is the mixed current-flux linkages space vector model which realises a the connection between the magnetising current space-vector and the rotor flux linkage space vector, selected as state-space variables $x = \left[i_{\mathrm{qm}}, i_{\mathrm{dm}}, \lambda'_{\mathrm{qr}}, \lambda'_{\mathrm{dr}} \right]^{\mathrm{T}}$ As the output vector components cannot be measured, this model, like the previous one, is prohibitive for implementation in vector control strategies. However, from computational effort, the state matrix contains the same number of zero elements (four) and the instantaneous electromagnetic torque is determined in a suitable form comparable to the classical vector control implementations.

$$
\frac{p}{\omega_b}\begin{bmatrix} i_{qm} \\ i_{dm} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} = \begin{bmatrix} \dfrac{r'_r x_{ls} x_m - r_s x'_r x'_{lr}}{x'_{lr} D} & 0 & \dfrac{r_s x'_{lr} - r'_r x_{ls}}{x'_{lr} D} & \dfrac{x_{ls}}{D} \cdot \dfrac{\omega_r}{\omega_b} \\ 0 & \dfrac{r'_r x_{ls} x_m - r_s x'_r x'_{lr}}{x'_{lr} D} & -\dfrac{x_{ls}}{D} \cdot \dfrac{\omega_r}{\omega_b} & \dfrac{r_s x'_{lr} - r'_r x_{ls}}{x'_{lr} D} \\ \dfrac{r'_r x_m}{x'_{lr}} & 0 & -\dfrac{r'_r}{x'_{lr}} & \dfrac{\omega_r}{\omega_b} \\ 0 & \dfrac{r'_r x_m}{x'_{lr}} & -\dfrac{\omega_r}{\omega_b} & -\dfrac{r'_r}{x'_{lr}} \end{bmatrix} \cdot \begin{bmatrix} i_{qm} \\ i_{dm} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} \dfrac{x'_{lr}}{D} & 0 & \dfrac{x_{ls}}{D} & 0 \\ 0 & \dfrac{x'_{lr}}{D} & 0 & \dfrac{x_{ls}}{D} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}
\tag{223}
$$

The electromagnetic torque is determined as follows:

$$
T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b x'_{lr}} \cdot \left(i_{qm} \lambda'_{dr} - i_{dm} \lambda'_{qr} \right)
\tag{224}
$$

2.9. Vector control startegies for three-phase induction machine

The aim of vector control is usually to decouple the stator current i_s into its flux producing and torque producing components (i_{ds}, i_{qs} respectively) in order to obtain a decoupled control of the flux and the electromagnetic torque. For this reason a special reference frame is selected fixed to different space vector variables. The reference frame has to be synchronous, as all the space vectors have the same angular velocity given by the supply voltage frequency (Vas - 1990, 1992), (Slemon - 1994), (Kelemen and Imecs - 1987).

Generally, the term of vector control is associated with field orientation control. This means that the special synchronous reference frame is linked to one of the flux linkages space vectors. The original field orientation sheme, developed more than twenty five years ago, was based on the alignement of the reference frame to the rotor flux. After 1985, (DeDoncker and Nowotny - 1988), (Erdman and Hoft - 1990) this control strategy was extented to the air-gap flux and to the stator flux alignement of the synchronuous reference frame. However, for vector control schemes there are two more possibilities, related to the currents space vectors: stator and rotor currents. All of these control strategies are investigated in this chapter, according to the modelling point of view of the induction machine. An easy to follow steps algorithm for implementing vector oriented control systems is obtained as follows (Lai - 1999):

1. A complete mathematical model of the three-phase induction machine is developed in stationary reference frame, according to the chosen set of state-space variables;
2. The rotor based variables are completely expressed in the new state-variable system;
3. The rotor angular velocity ω_r term is substituted with (ω_r - ω_k) where ω_k is the angular velocity of the synchronous frame;
4. The synchronous reference frame is selected linked to one of the space vectors, which means that the q-axis component of the reference space vector is null;
5. The torque equation is computed according to the selected flux or current space vector in the synchronous reference frame.

The transformation of reference frames for the induction motor vector control can be summarised as shown in Fig. 2.18.

Fig. 2.18. Block diagram of transformation of frames and co-ordinate systems
for induction motor vector control

2.9.1. Stator flux field orientation (SFO)

For this vector control strategy, the set of state-space variables formed by stator flux linkage and current space vectors $x = \left[\lambda_{qs}, \lambda_{ds}, i_{qs}, i_{ds} \right]^{\mathrm{T}}$ is selected. The mathematical model is given in the chapter dedicated to d-q models of the three-phase induction machine.

The rotor based variables expressed in state-space variables terms are:

$$i_{qr}^{\prime s} = \frac{1}{x_m}\left(\lambda_{qs}^s - x_s i_{qs}^s \right)$$

$$i_{dr}^{\prime s} = \frac{1}{x_m}\left(\lambda_{ds}^s - x_s i_{ds}^s \right)$$

$$\lambda_{qr}^{\prime s} = \frac{1}{x_m}\left(x_r^{\prime} \lambda_{qs}^s - D \cdot i_{qs}^s \right)$$

$$\lambda_{dr}^{\prime s} = \frac{1}{x_m}\left(x_r^{\prime} \lambda_{ds}^s - D \cdot i_{ds}^s \right)$$

(225-228)

To obtain the machine equations in the synchronuous stator flux reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the stator flux to be zero. The stator voltages equations remain unchanged. If a cage rotor is considered, the resultant equations are as follows:

$$\left(r_r^{\prime} x_s + D\frac{p}{\omega_b} \right) i_{qs}^s - \frac{\omega_s}{\omega_b}\left(x_r^{\prime} \lambda_{ds}^s - D \cdot i_{ds}^s \right) - x_r^{\prime} \frac{p}{\omega_b} \lambda_{qs}^s = 0$$

$$\left(r_r^{\prime} x_s + D\frac{p}{\omega_b} \right) i_{ds}^s - \frac{\omega_s}{\omega_b} D \cdot i_{qs}^s = \left(r_r^{\prime} + x_r^{\prime}\frac{p}{\omega_b} \right)\lambda_{ds}^s - \frac{\omega_s}{\omega_b} x_r^{\prime} \lambda_{qs}^s$$

(229-230)

where the definition used for the slip speed is: $\omega_s = \omega_e - \omega_r$.

If the special reference frame is fixed to the stator flux linkage vector, the q-component of this flux vector is defined equal to zero:

$$\lambda_{qs}^s = 0$$

$$\lambda_s^s = \lambda_{ds}^s$$

(231)

From stator flux linkage equations, the q-current components are given by:

$$i^s_{qs} = i^s_{qs}$$

$$i^s_{qr} = -\frac{x_s}{x_m} i^s_{qs}$$

(232)

The electromagnetic torque relation and slip speed can be derived in stator field orientation control as:

$$T_e = \frac{3P}{4} \cdot \frac{1}{\omega_b} \lambda^s_{ds} i^s_{qs}$$

$$\omega_s = \frac{(\omega_b r'_r x_s + D \cdot p) i^s_{qs}}{x'_r \lambda^s_{ds} - D \cdot i^s_{ds}}$$

(233-234)

The second dynamic equation of the machine, shows that there is a coupling between the stator current components. Consequently, any change in the torque producing component i^s_{qs} without changing i^s_{ds} accordingly will cause a transient in the stator flux. A decoupler is necessary to overcome this disadvantage. Therefore the command current of the *d*-axis component of the stator current can be calculated as follows:

$$i^s_{ds} = \left(K_p + \frac{K_i}{p} \right) \Delta \lambda^s_{ds} + i^s_{dq}$$

$$i^s_{dq} = \frac{\omega_s D \cdot i^s_{qs}}{\omega_b r'_r x_s + D \cdot p}$$

(235-236)

where K_p and K_i are proportional, respectively integral coefficients of the flux controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network) as demonstrated by Xu and Nowotny (1988).

Steady-state perfomance of a stator flux oriented system

Letting the derivative operator $p = 0$, one can obtain the steady-state voltage equations of the induction machine. After several manipulations of the system equations, the *d* current components are given by:

$$i^s_{ds} = \frac{\omega_b r'_r}{\omega_s} \cdot \frac{x_s}{x^2_m} i^s_{qs} + \frac{x'_r}{x_m} \left(\frac{x_s x'_r - x^2_m}{x_m} \right) \frac{\omega_s}{\omega_b r'_r} i^s_{qs}$$

$$i^s_{dr} = -\left(\frac{x_s x'_r - x^2_m}{x_m} \right) \frac{\omega_s}{\omega_b r'_r} i^s_{qs}$$

(237)

it yields the slip speed equation:

$$\omega^2_s i^s_{qs} D^2 - \omega_s \omega_b r'_r x^2_m \lambda^s_s + (\omega_b r'_r)^2 x^2_s i^s_{qs} = 0$$

(238)

The solutions of the above equations have to be real, for a given stator flux linkage. This means that the determinant of the second order equation satisfy the condition:

$$\Delta = \left(x^2_m \omega_b r'_r \lambda^s_s \right)^2 - 4D^2 (\omega_b r'_r x_s)^2 (i^s_{qs})^2 \geq 0$$

(239)

The maximum values for the *q* component of the stator current, the slip speed and the electromagnetic torque (pull-out torque) are:

$$\left(i_{qs}^{s} \right)_{max} = \frac{x_{m}^{2} \lambda_{s}^{s}}{2 D x_{s}}$$

$$\left(\omega_{s} \right)_{max} = \frac{r_{r}' x_{s}}{D}$$

$$\left(T_{e} \right)_{max} = \frac{3P}{4} \cdot \frac{x_{m}^{2} \left(\lambda_{s}^{s} \right)^{2}}{2 \omega_{b} D x_{s}}$$

(240-242)

If the angular slip velocity is larger than $\left(\omega_{s} \right)_{max}$ static instability will arise. This maximum value depends only on machine parameters and not on stator flux level. Neveretheless, the pull-out torque depends on the square of the flux magnitude. The pull-out torque determines the limit for the system stability operation zone, when the stator flux oriented control is employed. It is possible to limit the torque command not to exceed the pull out torque for a given stator flux. Some important conclusions cand be drawn for this vector control strategy:

- The electromagnetic torque and stator flux producing components of the stator current are not decoupled;
- A parameter dependent decoupling network has to be included;
- Torque and flux control does not require speed feed back;
- For operation at low speed it is difficult to estimate the stator flux;
- It is a good alternative for medium performance drive.

2.9.2. Rotor flux field orientation (RFO)

For this vector control strategy, the set of state-space variables formed by rotor flux linkage and stator current space vectors $\boldsymbol{x} = \left[\lambda'_{qr}, \lambda'_{dr}, i_{qs}, i_{ds} \right]^{T}$ is selected. If the induction motor is equipped with wound rotor, as the rotor currents are also measurable, the rotor current space vector can be also selected as state-space variables together with the rotor flux linkage space vector $\boldsymbol{x} = \left[\lambda'_{qr}, \lambda'_{dr}, i'_{qr}, i'_{dr} \right]^{T}$. The mathematical model is given in the chapter dedicated to d-q models of the three-phase induction machine. The first case will be analysed, for the second one the algoritm is similar.

The stator flux linkage and rotor current space vectors components expressed as functions in terms of state-space variables are:

$$\lambda_{qs}^{r} = \frac{1}{x_{r}'} \left(x_{m} \lambda_{qr}^{\prime r} + D \cdot i_{qs}^{r} \right)$$

$$\lambda_{ds}^{r} = \frac{1}{x_{r}'} \left(x_{m} \lambda_{dr}^{\prime r} + D \cdot i_{ds}^{r} \right)$$

$$i_{qr}^{\prime r} = \frac{1}{x_{r}'} \left(\lambda_{qr}^{\prime r} - x_{m} i_{qs}^{r} \right)$$

$$i_{dr}^{\prime r} = \frac{1}{x_{r}'} \left(\lambda_{dr}^{\prime r} - x_{m} i_{ds}^{r} \right)$$

(243-246)

The voltage equations re-written in terms of the state-space variables become:

$$u_{qs}^{r} = \left(r_s + \frac{D}{x'_r} \cdot \frac{p}{\omega_b} \right) i_{qs}^{r} + \frac{\omega_s}{\omega_b} \cdot \frac{D}{x'_r} i_{ds}^{r} + x_m \frac{p}{\omega_b} \lambda_{qr}^{'r} + x_m \frac{\omega_s}{\omega_b} \lambda_{dr}^{''r}$$

$$u_{ds}^{r} = \left(r_s + \frac{D}{x'_r} \cdot \frac{p}{\omega_b} \right) i_{ds}^{r} - \frac{\omega_s}{\omega_b} \cdot \frac{D}{x'_r} i_{qs}^{r} + x_m \frac{p}{\omega_b} \lambda_{dr}^{''r} - x_m \frac{\omega_s}{\omega_b} \lambda_{qr}^{''r}$$

$$\text{(247-250)}$$

$$u'^{r}_{qr} = 0 = -x_m \frac{r'_r}{x'_r} i_{qs}^{r} + \left(\frac{r'_r}{x'_r} + \frac{p}{\omega_b} \right) \lambda_{qr}^{''r} + \frac{\omega_s}{\omega_b} \lambda_{dr}^{''r}$$

$$u'^{r}_{dr} = 0 = -x_m \frac{r'_r}{x'_r} i_{ds}^{r} + \left(\frac{r'_r}{x'_r} + \frac{p}{\omega_b} \right) \lambda_{dr}^{''r} - \frac{\omega_s}{\omega_b} \lambda_{qr}^{''r}$$

If the special reference frame is fixed to the rotor flux linkage vector, the q-component of this flux vector is defined equal to zero:

$$\lambda_{qr}^{''r} = 0$$

$$\lambda_{r}^{''r} = \lambda_{dr}^{''r}$$

$$\text{(251)}$$

From stator flux linkage equations, the q-current components are given by:

$$i_{qs}^{r} = i_{qs}^{r}$$

$$i_{qr}^{''r} = -\frac{x_m}{x_r} i_{qs}^{r}$$

$$\text{(252)}$$

The flux producing component of the stator current is determined as follows:

$$i_{ds}^{r} = \frac{\left(1 + \frac{x'_r}{r'_r} \cdot \frac{p}{\omega_b} \right) \lambda_{dr}^{''r}}{x_m}$$

$$\text{(253)}$$

The above relation shows that there is no need of a current decoupler in rotor field orientation scheme. Both stator current components (torque and flux producing) can be controlled independently.

Steady-state perfomance of a rotor flux oriented system

Letting the derivative operator $p = 0$, one can obtain the steady-state voltage equations of the induction machine. After several manipulations of the system equations, the d current components are given by:

$$i_{ds}^{r} = \frac{\lambda_{dr}^{''r}}{x_m} = \frac{\omega_b r'_r}{\omega_s} \cdot \frac{i_{qs}^{r}}{x'_r}$$

$$i_{dr}^{r} = 0$$

$$\text{(254)}$$

It yields the slip speed equation:

$$\omega_s \lambda_{dr}^{''r} - \frac{x_m}{x'_r} \cdot \omega_b r'_r i_{qs}^{r} = 0$$

$$\text{(255)}$$

which for a given rotor flux has always real solution. Thus the resulting current controlled slip speed and the electromagnetic torque are:

$$\omega_s = \frac{x_m r'_r \omega_b i_{qs}^{r}}{x'_r \lambda_{dr}^{''r}}$$

$$\text{(256-257)}$$

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{x'_r \omega_b} \lambda_{dr}^{''r} i_{qs}^{r} = \frac{3P}{4} \cdot \frac{x_m^2}{\left(x'_r \right)^2} \left(i_{qs}^{r} \right)^2 \cdot \frac{r'_r}{\omega_s}$$

The most important features of rotor field oriented (RFO) vector control are:

- As it is the original vector control strategy developed by Blashke (1971), the RFO represents the most popular approach in vector control strategies;
- It provides complete decoupling of the torque and flux producing components of the stator current;
- There are possible *direct vector control strategy* (the usage of sensors or model to provide feedback of the flux magnitude and orientation) and *indirect vector control strategy* (the usage of assumed slip frequency relationship to achieve field orientation);
- The direct vector control strategy is the optimum choice for medium and high-speed applications;
- The indirect vector control strategy is the optimum choice for low-speed applications;

2.9.3. Air-gap flux field orientation (AFO)

For this vector control strategy, the set of state-space variables formed by air-gap flux linkage and stator current space vectors $x = \left[\lambda_{qm}, \lambda_{dm}, i_{qs}, i_{ds} \right]^{T}$ is selected. If the induction motor is equipped with wound rotor, as the rotor currents are also measurable, the rotor current space vector can be also selected as state-space variables together with the air-gap flux linkage space vector $x = \left[\lambda_{qm}, \lambda_{dm}, i'_{qr}, i'_{dr} \right]^{T}$. The mathematical model is given in the chapter dedicated to d-q models of the three-phase induction machine. As both cases are similar to analyse, only the first case will be detailed.

The stator and rotor flux linkages and rotor current space vectors components expressed as functions in terms of state-space variables are:

$$\lambda_{qs}^{m} = \lambda_{qm}^{m} + \left(x_{s} - x_{m} \right) i_{qs}^{m}$$

$$\lambda_{ds}^{m} = \lambda_{dm}^{m} + \left(x_{s} - x_{m} \right) i_{ds}^{m}$$

$$\lambda'^{m}_{qr} = \frac{x'_{r}}{x_{m}} \lambda_{qm}^{m} - \left(x'_{r} - x_{m} \right) i_{qs}^{m} \tag{258-261}$$

$$\lambda_{qs}^{m} = \frac{x'_{r}}{x_{m}} \lambda_{qm}^{m} - \left(x'_{r} - x_{m} \right) i_{ds}^{m}$$

$$i'^{r}_{qr} = \frac{1}{x_{m}} \lambda_{qm}^{m} - i_{qs}^{m}$$

$$\tag{262-263}$$

$$i'^{r}_{dr} = \frac{1}{x_{m}} \lambda_{dm}^{m} - i_{ds}^{m}$$

The voltage equations of the machine expressed in state-variables terms, are readily deductible now:

$$u_{qs}^{m} = \left(r_{s} + (x_{s} - x_{m}) \frac{p}{\omega_{b}} \right) i_{qs}^{m} + \frac{p}{\omega_{b}} \lambda_{qm}^{m} + \frac{\omega_{s}}{\omega_{b}} \left(\lambda_{dm}^{m} + (x_{s} - x_{m}) i_{ds}^{m} \right)$$

$$u_{ds}^{m} = \left(r_{s} + (x_{s} - x_{m}) \frac{p}{\omega_{b}} \right) i_{ds}^{m} + \frac{p}{\omega_{b}} \lambda_{dm}^{m} - \frac{\omega_{s}}{\omega_{b}} \left(\lambda_{qm}^{m} + (x_{s} - x_{m}) i_{qs}^{m} \right)$$

$$\tag{264-267}$$

$$u'^{m}_{qr} = 0 = - \left(r'_{r} + (x'_{r} - x_{m}) \frac{p}{\omega_{b}} \right) i_{qs}^{m} + \left(\frac{r'_{r}}{x_{m}} + \frac{x'_{r}}{x_{m}} \cdot \frac{p}{\omega_{b}} \right) \lambda_{qm}^{m} + \frac{\omega_{s}}{\omega_{b}} \left(\frac{x'_{r}}{x_{m}} \lambda_{dm}^{m} - (x'_{r} - x_{m}) i_{ds}^{m} \right)$$

$$u'^{m}_{dr} = 0 = - \left(r'_{r} + (x'_{r} - x_{m}) \frac{p}{\omega_{b}} \right) i_{ds}^{m} + \left(\frac{r'_{r}}{x_{m}} + \frac{x'_{r}}{x_{m}} \cdot \frac{p}{\omega_{b}} \right) \lambda_{dm}^{m} - \frac{\omega_{s}}{\omega_{b}} \left(\frac{x'_{r}}{x_{m}} \lambda_{qm}^{m} - (x'_{r} - x_{m}) i_{qs}^{m} \right)$$

If the special reference frame is fixed to the air-gap flux linkage vector, the *q*-component of this flux vector is defined equal to zero:

$$\lambda_{qm}^{m} = 0$$

$$\lambda_{m}^{m} = \lambda_{dm}^{m}$$

(268)

From air-gap flux linkage equations, the q-current components are given by:

$$i_{qs}^{m} = i_{qs}^{m}$$

$$i_{qr}^{m} = -i_{qs}^{m}$$

(269)

The flux producing component of the stator current is determined as follows:

$$\left(r'_{r} + (x'_{r} - x_{m})\frac{p}{\omega_{b}}\right)i_{ds}^{m} = \left(\frac{r'_{r}}{x_{m}} + \frac{x'_{r}}{x_{m}}\cdot\frac{p}{\omega_{b}}\right)\lambda_{dm}^{m} + (x'_{r} - x_{m})\frac{\omega_{s}}{\omega_{b}}i_{qs}^{m}$$

(270)

One can note that the d-axis component of stator current for AFO is not only controllled by the *d*-axis air-gap flux componennt, but also by the *q*-axis (torque producing) component of the stator current. It is necessary to decouple the stator current components, in order to achieve a linear control. For this reason, the command of current of the *d*-axis component is computed as follows:

$$i_{ds}^{m} = \left(K_{p} + \frac{K_{i}}{p}\right)\Delta\lambda_{dm}^{m} + i_{dq}^{m}$$

$$i_{dq}^{m} = \frac{\omega_{s}(x'_{r} - x_{m})\cdot i_{qs}^{m}}{\omega_{b}r'_{r} + (x'_{r} - x_{m})\cdot p}$$

(271-272)

The electromagnetic torque relation and slip speed can be derived in air-gap field orientation control as:

$$T_{e} = \frac{3P}{4}\cdot\frac{1}{\omega_{b}}\lambda_{dm}^{m}i_{qs}^{m}$$

$$\omega_{s} = \frac{[\omega_{b}r'_{r} + (x'_{r} - x_{m})\cdot p]i_{qs}^{m}}{\frac{x'_{r}}{x_{m}}\lambda_{dm}^{m} - (x'_{r} - x_{m})\cdot i_{ds}^{m}}$$

(273-274)

Steady-state perfomance of an air-gap flux oriented system

Letting the derivative operator *p* =0, the steady-state voltage equations of the induction machine are readily determined. After several manipulations of the system equations, the *d* current components are given by:

$$i_{ds}^{m} = \frac{r'_{r}}{x_{m}}\cdot\frac{\omega_{b}}{\omega_{s}}i_{qs}^{m} + \frac{x'_{r}}{x_{m}}(x'_{r} - x_{m})\cdot\frac{\omega_{s}}{\omega_{b}}i_{qs}^{m}$$

$$i_{dr}^{m} = \frac{(x_{m} - x'_{r})i_{qs}^{m}}{r'_{r}}\cdot\frac{\omega_{s}}{\omega_{b}}$$

(275)

It yields the slip speed equation:

$$\omega_{s}^{2}\cdot\frac{(x'_{r} - x_{m})^{2}}{r'_{r}\omega_{b}}\cdot i_{qs}^{m} - \omega_{s}\lambda_{dm}^{m} + r'_{r}\omega_{b}i_{qs}^{m} = 0$$

(276)

The solutions of the above equations have to be real, for a given stator flux linkage. This means that the determinant of the second order equation satisfy the condition:

$$\Delta = \left(\lambda_{dm}^{m}\right)^{2} - 4\left(x'_{r} - x_{m}\right)^{2}\left(i_{qs}^{m}\right)^{2} \geq 0$$

(277)

The maximum values for the q component of the stator current, the slip speed and the electromagnetic torque (pull-out torque) are:

$$\left(i_{qs}^{m}\right)_{max} = \frac{\lambda_{dm}^{m}}{2(x'_{r} - x_{m})}$$

$$\left(\omega_{s}\right)_{max} = \frac{r'_{r}}{x'_{r} - x_{m}}$$

$$\left(T_{e}\right)_{max} = \frac{3P}{8} \cdot \frac{\left(\lambda_{dm}^{m}\right)^{2}}{x'_{r} - x_{m}}$$

(278-280)

When the angular slip velocity is larger than $\left(\omega_{s}\right)_{max}$ instability will occur. One should note that the maximum (pull-out) angular slip velocity depends only on the rotor parameters and does not depends on the air-gap flux. However, the maximum (pull-out) torque is proportional to the square of the air-gap flux magnitude and thus a small increment of air-gap flux will detetmine a significant increment of the electromagnetic torque.

Some important conclusions cand be drawn for the air-gap field orientation (AFO) control strategy:

- The electromagnetic torque and air-gap flux producing components of the stator current are not decoupled;
- A parameter dependent decoupling network has to be included;
- Torque and flux control does not require speed feed back;
- No sophisticated parameter estimation methods or model based observers is required;
- It is a good alternative for low and medium performance drives as the air-gap flux can be measured directly;

2.9.4. Stator current orientation (SCO)

If the field orientation control is well established with multiple practical solutions for different industrial applications, an unconventional method of controlling the speed and torque for the induction machine, is given by the selection of currents space vectors as linking basis for the synchronuous reference frame. The set of state-space variables is identical with the stator flux field orientation control (SFO): the stator flux and current space vectors $x = \left[\lambda_{qs}, \lambda_{ds}, i_{qs}, i_{ds}\right]^{T}$. The mathematical model is identical to that used in SFO case.

The rotor based variables expressed in state-space variables terms are:

$$i_{qr}^{sc} = \frac{1}{x_{m}}\left(\lambda_{qs}^{sc} - x_{s}i_{qs}^{sc}\right)$$

$$i_{dr}^{sc} = \frac{1}{x_{m}}\left(\lambda_{ds}^{sc} - x_{s}i_{ds}^{sc}\right)$$

(281-284)

$$\lambda_{qr}^{sc} = \frac{1}{x_{m}}\left(x'_{r}\lambda_{qs}^{sc} - D \cdot i_{qs}^{sc}\right)$$

$$\lambda_{dr}^{sc} = \frac{1}{x_{m}}\left(x'_{r}\lambda_{ds}^{sc} - D \cdot i_{ds}^{sc}\right)$$

To obtain the machine equations in the synchronuous stator current reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the stator current to be zero. The stator voltages equations remain unchanged. If a cage rotor is considered, the resultant equations are as follows:

$$\left(r'_r\, x_s + D\,\frac{p}{\omega_b} \right) i_{qs}^{sc} - \frac{\omega_s}{\omega_b}\left(x'_r\, \lambda_{ds}^{sc} - D\cdot i_{ds}^{sc} \right) - \left(r'_r + x'_r\,\frac{p}{\omega_b} \right)\lambda_{qs}^{sc} = 0$$

$$\left(r'_r\, x_s + D\,\frac{p}{\omega_b} \right) i_{ds}^{sc} - \frac{\omega_s}{\omega_b} D\cdot i_{qs}^{sc} - \left(r'_r + x'_r\,\frac{p}{\omega_b} \right)\lambda_{ds}^{sc} + \frac{\omega_s}{\omega_b} x'_r\,\lambda_{qs}^{sc} = 0 \tag{285-286}$$

If the special reference frame is fixed to the stator current space vector, the *q*-component of this current vector is defined equal to zero:

$$i_{qs}^{sc} = 0$$

$$i_s^s = i_{ds}^{sc} \tag{287}$$

Different from the SFO case, we have to express the *q*-flux components, by considering the flux linkages equations:

$$\lambda_{qs}^{sc} = \lambda_{qs}^{sc}$$

$$\lambda'^{sc}_{qr} = \frac{x'_r}{x_m}\lambda_{qs}^{sc} \tag{288}$$

The electromagnetic torque relation and slip speed can be derived in stator current orientation control as:

$$T_e = -\frac{3P}{4}\cdot\frac{1}{\omega_b}\,\lambda_{qs}^{sc} i_{ds}^{sc}$$

$$\omega_s = \frac{\left(\omega_b r'_r\, x_s + x'_r\cdot p\right)\lambda_{qs}^{sc}}{D\cdot i_{ds}^s - x'_r\,\lambda_{ds}^{sc}} \tag{289-290}$$

From the second dynamic equation of the machine, a relation between the d-axis component of the stator current and stator flux linkage space vector components can be deduced:

$$\left(r'_r\, x_s + D\,\frac{p}{\omega_b} \right) i_{ds}^{sc} - \left(r'_r + x'_r\,\frac{p}{\omega_b} \right)\lambda_{ds}^{sc} + \frac{\omega_s}{\omega_b} x'_r\,\lambda_{qs}^{sc} = 0 \tag{291}$$

The previous dynamic equation of the machine, shows that there is a coupling between the stator flux linkage components. Consequently, any change in the torque producing component λ_{ds}^{sc} without changing λ_{qs}^{sc} accordingly will cause a trensient in the stator flux. A decoupler is necessary to overcome this disadvantage. Therefore the command current of the *d*-axis component of the stator current can be calculated as follows:

$$\lambda_{ds}^{sc} = \left(K_p + \frac{K_i}{p} \right)\Delta i_{ds}^{sc} + \lambda_{dq}^{sc}$$

$$\lambda_{dq}^{sc} = \frac{\dfrac{\omega_s}{\omega_b} x'_r\,\lambda_{qs}^{sc}}{\left(r'_r + x'_r\,\dfrac{p}{\omega_b} \right)} \tag{291-292}$$

where K_p and K_i are proportional, respectively integral coefficients of the current controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance of a stator current oriented system

Letting the derivative operator $p = 0$, one can obtain the steady-state voltage equations of the induction machine. After several manipulations of the system equations, the *d* current components are given by:

$$\lambda_{ds}^{sc} = -\lambda_{qs}^{sc} \cdot \frac{x'_r}{r'_r} \cdot \frac{D}{x_m^2} \cdot \frac{\omega_s}{\omega_b} - \lambda_{qs}^{sc} \cdot \frac{x_s r'_r}{x_m^2} \cdot \frac{\omega_b}{\omega_s}$$

$$\lambda_{dr}^{sc} = -\frac{r'_r}{x_m} \cdot \frac{\omega_b}{\omega_s} \lambda_{qs}^{sc}$$

(293)

it yields the slip speed equation:

$$\left(\frac{\omega_s}{\omega_b}\right)^2 \left(\frac{x'_r}{r'_r}\right) \cdot \frac{1}{x_m} \lambda_{qs}^{sc} + \left(\frac{\omega_s}{\omega_b}\right) \cdot \frac{1}{r'_r} i_{ds}^{sc} + \frac{1}{x_m^2} \lambda_{qs}^{sc} = 0$$

(294)

The solutions of the above equations have to be real, for a given stator current. This means that the determinant of the second order equation satisfy the condition:

$$\Delta = \left(\frac{i_{ds}^{sc}}{r'_r}\right)^2 - 4\left(\frac{x'_r}{r'_r}\right)^2 \left(\frac{\lambda_{qs}^{sc}}{x_m^2}\right)^2 \geq 0$$

(295)

The maximum values for the q component of the stator flux linkage, the slip speed and the electromagnetic torque (pull-out torque) are:

$$\left(\lambda_{qs}^{sc}\right)_{max} = \frac{x_m^2 i_{ds}^{sc}}{2x'_r}$$

$$\left(\omega_s\right)_{max} = \frac{r'_r}{x'_r}$$

(296-299)

$$\left(T_e\right)_{max} = \frac{3P}{4} \cdot \frac{x_m^2 \left(i_{ds}^{sc}\right)^2}{2\omega_b x'_r}$$

If the angular slip velocity is larger than $\left(\omega_s\right)_{max}$ static instability will arise. This maximum value depends only on machine parameters and not on stator current or flux level. Nevertheless, the pull-out torque depends on the square of the stator current magnitude. The pull-out torque determines the limit for the system stability operation zone, when the stator current space vector oriented control is employed. It is possible to limit the torque command not to exceed the pull out torque for a given stator current. Some important conclusions can be drawn for the stator current orientation (SCO) vector control strategy:

- The electromagnetic torque and stator current related components of the stator flux linkage are not decoupled;
- A parameter dependent decoupling network has to be included;
- Torque and flux control does not require speed feed back;
- An operation at low speed is easy to realise as there is no need to estimate the stator flux;
- It is a good alternative for wide range speed drives.

2.9.5. Rotor current orientation (RCO)

Another unconventional vector control strategy is the one wich comprises a synchronuous reference frame linked to the rotor current space vector. The rotor current orientation control strategy (RCO) is analysed for a cage rotor induction motor. The selected set of state-space variables is given by two options if we consider the criteria of direct measurable quantities:

I) The stator flux linkage and rotor current space vectors: $x = \left[\lambda_{qs}, \lambda_{ds}, i'_{qr}, i'_{dr}\right]^T$;

When the stator flux and and the rotor current are selected as state-space variables, one can derive the rotor flux and stator current functions in terms of state variables as follows:

$$i_{qs}^{rc} = \frac{1}{x_s}\left(\lambda_{qs}^{rc} - x_m i\,'_{qr}^{\,rc}\right)$$

$$i_{ds}^{rc} = \frac{1}{x_s}\left(\lambda_{ds}^{rc} - x_m i\,'_{dr}^{\,rc}\right)$$

$$\lambda\,'_{qr}^{\,rc} = \frac{x_m}{x_s}\lambda_{qs}^{rc} + \frac{D}{x_s}i\,'_{qr}^{\,rc}$$

$$\lambda\,'_{dr}^{\,rc} = \frac{x_m}{x_s}\lambda_{ds}^{rc} + \frac{D}{x_s}i\,'_{dr}^{\,rc}$$

(300-304)

To obtain the machine equations in the synchronuous rotor current reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the rotor current to be zero. The stator voltages equations are re-written also. The resultant equations have the following form:

$$u_{qs}^{rc} = -\frac{r_s x_m}{x_s}i\,'_{qr}^{\,rc} + \left(\frac{r_s}{x_s} + \frac{p}{\omega_b}\right)\lambda_{qs}^{rc} + \frac{\omega_s}{\omega_b}\lambda_{ds}^{rc}$$

$$u_{ds}^{rc} = -\frac{r_s x_m}{x_s}i\,'_{dr}^{\,rc} + \left(\frac{r_s}{x_s} + \frac{p}{\omega_b}\right)\lambda_{ds}^{rc} - \frac{\omega_s}{\omega_b}\lambda_{qs}^{rc}$$

(305-308)

$$0 = \left(r'_r + \frac{D}{x_s}\cdot\frac{p}{\omega_b}\right)i\,'_{qr}^{\,rc} + \frac{D}{x_s}\cdot\frac{\omega_s}{\omega_b}i\,'_{dr}^{\,rc} + \frac{x_m}{x_s}\cdot\frac{p}{\omega_b}\lambda_{qs}^{rc} + \frac{x_m}{x_s}\cdot\frac{\omega_s}{\omega_b}\lambda_{ds}^{rc}$$

$$0 = \left(r'_r + \frac{D}{x_s}\cdot\frac{p}{\omega_b}\right)i\,'_{dr}^{\,rc} - \frac{D}{x_s}\cdot\frac{\omega_s}{\omega_b}i\,'_{qr}^{\,rc} + \frac{x_m}{x_s}\cdot\frac{p}{\omega_b}\lambda_{ds}^{rc} - \frac{x_m}{x_s}\cdot\frac{\omega_s}{\omega_b}\lambda_{qs}^{rc}$$

If the special reference frame is fixed to the rotor current space vector, the q-component of this current vector is defined equal to zero:

$$i\,'_{qr}^{\,rc} = 0$$

$$i_r^{rc} = i\,'_{dr}^{\,rc}$$

(309)

Different from the RFO case, we have to express the q-flux components, by considering the flux linkages equations:

$$\lambda_{qs}^{rc} = \lambda_{qs}^{rc}$$

$$\lambda\,'_{qr}^{\,rc} = \frac{x_s}{x_m}\lambda_{qs}^{sc}$$

(310)

The electromagnetic torque relation and slip speed can be derived in stator current orientation control as:

$$T_e = \frac{3P}{4}\cdot\frac{x_m}{x_s\omega_b}\lambda_{qs}^{rc}i\,'_{dr}^{\,rc}$$

$$\omega_s = \frac{-x_m\cdot p\,\lambda_{qs}^{rc}}{D\cdot i\,'_{dr}^{\,rc} + x_m\lambda_{ds}^{rc}}$$

(311-312)

From the second dynamic equation of the machine, a relation between the d-axis component of the rotor current and stator flux linkage space vector components can be deduced:

$$0 = \left(r'_r + \frac{D}{x_s}\cdot\frac{p}{\omega_b}\right)i\,'_{dr}^{\,rc} + \frac{x_m}{x_s}\cdot\frac{p}{\omega_b}\lambda_{ds}^{rc} - \frac{x_m}{x_s}\cdot\frac{\omega_s}{\omega_b}\lambda_{qs}^{rc}$$

(313)

A coupling between the stator flux linkage components appears, as can be deduced from the previous dynamic equation of the machine. Consequently, any change in the torque producing

component λ_{ds}^{rc} without changing λ_{qs}^{rc} accordingly, will cause a transient in the stator flux. The command current of the d-axis component of the stator current can be calculated in order to overcome this disadvantage:

$$\lambda_{ds}^{rc} = \left(K_p + \frac{K_i}{p} \right) \Delta i \, {}_{dr}^{\prime rc} + \lambda_{dq}^{rc}$$

$$\lambda_{dq}^{rc} = \frac{1}{p} \cdot \omega_s \lambda_{qs}^{sc}$$

(314-315)

where K_p and K_i are proportional, respectively integral coefficients of the current controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance of a rotor current oriented system

After several manipulations of the system equations if we let the derivative operator $p = 0$, the d current components are obtained from the steady-state voltage equations of the induction machine:

$$\lambda_{ds}^{rc} = -\frac{D}{x_s r'_r} \cdot \frac{\omega_s}{\omega_b} \lambda_{qs}^{rc}$$

$$\lambda \, {}_{dr}^{\prime rc} = 0$$

(316)

It yields the slip angular velocity equation:

$$i_{dr}^{rc} \cdot \frac{D}{x_s} - \frac{x_m D}{x_s^2 r'_r} \cdot \frac{\omega_s}{\omega_b} \lambda_{qs}^{rc} = 0$$

(317)

which for a given rotor flux has always real solution. Thus the slip speed and the electromagnetic torque are:

$$\omega_s = \frac{x_s r'_r \omega_b i \, {}_{dr}^{\prime rc}}{x_m \lambda_{qs}^{rc}}$$

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{x_s \omega_b} \lambda_{qs}^{rc} i \, {}_{dr}^{\prime rc}$$

(318-319)

The most important features of rotor current oriented (RCO) vector control with mixed flux and current state-space variables are:

• The electromagnetic torque and rotor current related components of the stator flux linkage are not decoupled;
• The decoupling network that has to be included is parameter independent and easy to implement;
• Torque and flux control require speed feed back;
• There is no stability limit as there is no pull-out slip speed or electromagnetic torque,
• An operation at low speed is easy to realise as there is no need to estimate the stator flux;
• It is a good alternative for wide range speed drives.

II) The stator and rotor currents $x = \left[i_{qs}, i_{ds}, i'_{qr}, i'_{dr} \right]^T$. When the stator and the rotor current are selected as state-space variables, one can derive the stator and rotor flux as functions in terms of state variables from the classical flux linkage equations.

To obtain the machine equations in the synchronuous rotor current reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the rotor current to be zero. The stator voltages equations are re-written also. The resultant equations have the following form:

$$u_{qs}^{rc} = \left(r_s + x_s \frac{p}{\omega_b} \right) i_{qs}^{rc} + x_m \frac{p}{\omega_b} i_{qr}^{\prime rc} + x_s \frac{\omega_s}{\omega_b} i_{ds}^{rc} + x_m \frac{\omega_s}{\omega_b} i_{dr}^{\prime rc}$$

$$u_{ds}^{rc} = -x_s \frac{\omega_s}{\omega_b} i_{qs}^{rc} - x_m \frac{\omega_s}{\omega_b} i_{qr}^{\prime rc} + \left(r_s + x_s \frac{p}{\omega_b} \right) i_{ds}^{rc} + x_m \frac{p}{\omega_b} i_{dr}^{\prime rc}$$

$$0 = x_m \frac{p}{\omega_b} i_{qs}^{rc} + \left(r'_r + x'_r \frac{p}{\omega_b} \right) i_{qr}^{\prime rc} + x_m \frac{\omega_s}{\omega_b} i_{ds}^{rc} + x'_r \frac{\omega_s}{\omega_b} i_{dr}^{\prime rc}$$

$$0 = -x_m \frac{\omega_s}{\omega_b} i_{qs}^{rc} - x'_r \frac{\omega_s}{\omega_b} i_{qr}^{\prime rc} + x_m \frac{p}{\omega_b} i_{ds}^{rc} + \left(r'_r + x'_r \frac{p}{\omega_b} \right) i_{dr}^{\prime rc}$$

(320-323)

If the special reference frame is fixed to the rotor current space vector, the *q*-component of this current vector is defined equal to zero:

$$i_{qr}^{\prime rc} = 0$$

$$i_r^{rc} = i_{dr}^{\prime rc}$$

(324)

Different from the (*I*) case of RCO control strategy, for the (*II*) case we have to express the stator flux linkage *q*-components, by considering the flux linkages equations:

$$\lambda_{qs}^{rc} = x_s i_{qs}^{rc}$$

$$\lambda_{qr}^{\prime rc} = x_m i_{qs}^{sc} = \frac{x_m}{x_s} \lambda_{qs}^{rc}$$

(325)

From the second dynamic equation of the machine, a relation between the *d*-axis component of the rotor current and stator current space vector components can be deduced:

$$x_m \frac{p}{\omega_b} i_{ds}^{rc} = x_m \frac{\omega_s}{\omega_b} i_{qs}^{rc} - \left(r'_r + x'_r \frac{p}{\omega_b} \right) i_{dr}^{\prime rc}$$

(326)

The electromagnetic torque relation and slip speed can be derived in stator current orientation control as:

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b} \cdot i_{qs}^{rc} i_{dr}^{\prime rc}$$

$$\omega_s = -\frac{x_m \cdot p i_{qs}^{rc}}{x_m i_{ds}^{rc} + x'_r i_{dr}^{\prime rc}}$$

(327-328)

A coupling between the *d*-axis and *q*-axis stator current components appears, as can be deduced from the previous dynamic equation of the machine. Consequently, any change in the torque producing component i_{ds}^{rc} without changing i_{qs}^{rc} accordingly, will cause a transient in the stator flux. The command current of the *d*-axis component of the stator current can be calculated in order to overcome this disadvantage:

$$i_{ds}^{rc} = \left(K_p + \frac{K_i}{p} \right) \Delta i_{dr}^{\prime rc} + i_{dq}^{rc}$$

$$i_{dq}^{rc} = \frac{1}{p} \cdot \omega_s i_{qs}^{sc}$$

(329-330)

where K_p and K_i are proportional, respectively integral coefficients of the current controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance of a rotor current oriented system

After several manipulations of the system equations if we let the derivative operator $p = 0$, the *d* flux components are obtained from the steady-state voltage equations of the induction machine:

$$\lambda_{ds}^{rc} = -\frac{D}{r'_r} \cdot \frac{\omega_s}{\omega_b} i_{qs}^{rc} \tag{331}$$

$$\lambda''^{rc}_{dr} = 0$$

It yields the slip angular velocity equation:

$$i_{dr}^{rc} \cdot \frac{D}{x_s} - \frac{x_m D}{x_s r'_r} \cdot \frac{\omega_s}{\omega_b} i_{qs}^{rc} = 0 \tag{332}$$

which for a given rotor flux has always real solution. Thus the slip speed and the electromagnetic torque are:

$$\omega_s = \frac{r'_r \omega_b i''^{rc}_{dr}}{x_m i_{qs}^{rc}}$$

$$T_e = \frac{3P}{4} \cdot \frac{x_m}{\omega_b} i_{qs}^{rc} i''^{rc}_{dr} \tag{333-334}$$

The rotor current oriented (RCO) vector control with currents state-space variables presents the same features as the mixed flux-current state variables model. However the current model is easier to be implemented as it provides directly the estimated values for the two-axis co-ordinate stator current. Thus the command for a current PWM inverter is readily obtainable. Also, as feedback measurements, the stator currents sensors are more reliable and cheaper than the flux sensors.

3. CONTINUOS-TIME DOMAIN LINEAR MODELS OF THE SINGLE- PHASE INDUCTION MACHINE

3.1. Introduction

The induction machine is used in a wide range of applications as an electrical to mechanical energy converter. The single-phase induction machine is the most used converter in home appliances.

The analysis of the induction machine is essentially the same for a three-phase, two-phase or single-phase machine. An accurate mathematical model for the induction machine is necessary to be determined in vector control operation. This model has to be suitable for the analysis of both the steady-state operation and the dynamic operation of the system. If the double-revolving field (Veinott - 1959), or the symmetrical component theories (Fortescue - 1918) permit a detailed simulation and modelling of the single-phase induction machine in steady-state operation, for the dynamic analysis of the machine a different mathematical approach has to be found.. Starting from the reference frame theory, with voltages, currents and fluxes referred to a two-axis quadrature co-ordinates system, a general model for the single-phase induction machine is developed according to Krause (1965) and Krause et al (1995). Different from the three-phase induction machine, where two approaches are valid (*d-q* axes and space vector theory), the single-phase induction machine is completely described only by a two-axis quadrature axis.

3.2. Voltage and flux-current equations of the single-phase induction machine

In Fig. 3.1 is illustrated a single-phase induction machine. The following assumptions have been made:
a) electrically orthogonal stator windings with sinusoidal distribution;
b) only the fundamental-space-harmonic-component of the air-gap flux distribution will be considered;
c) magnetic-saturation effects, core loss and stray load losses are negligible;
d) magnetic-diffusion (i.e. deep-bar) effects in the rotor will be ignored. This assumption is typically valid in small induction machine. It is further justified by the fact that under most operating conditions, the single-phase induction motor will be operating at low slips and hence the rotor currents will be at frequencies sufficiently low that magnetic-diffusion effects are insignificant.
e) temperature effect on windings resistance and reactance value is negligible;
f) the lamination magnetic permeability is considered infinite;
g) in steady-state operation, the voltages and currents are sinusoidal.
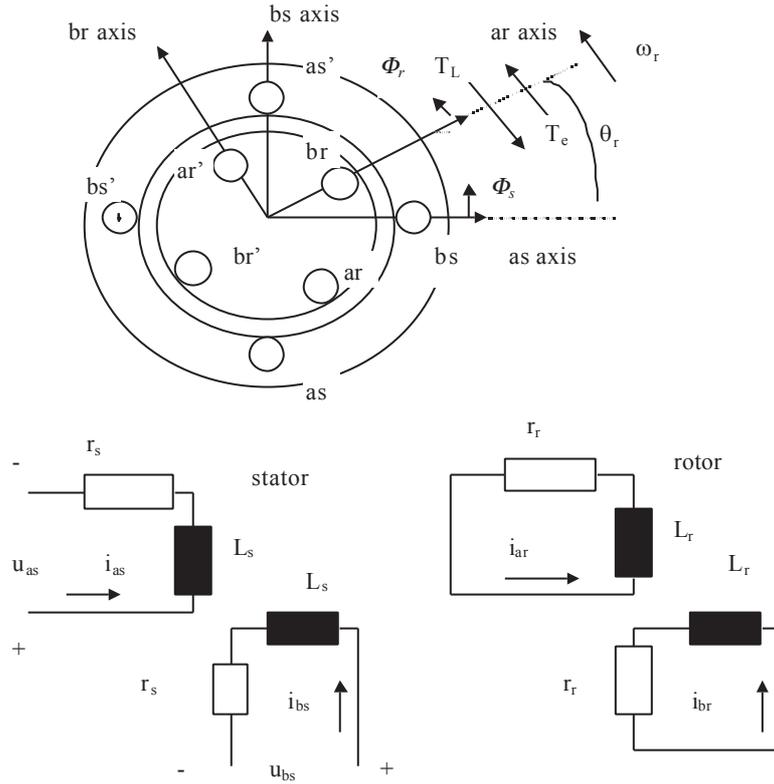
Fig.3.1. The real model of the symmetrical single-phase induction machine with squirrel-cage rotor represented by two identical windings

The machine rotor is described by two identical magnetically orthogonal windings. Considering an arbitrarily reference frame, the spatial position of the stator winding is characterised by the electrical angle Φ_s and the spatial position of the rotor winding is characterised by the electrical angle Φ_r. The angular speed of the rotor is ω_r and the displacement between stator and rotor windings is θ_r. These angles are linked through the relation:

$$\Phi_s = \Phi_r + \theta_r \tag{335}$$

The voltage equations related to the machine from Fig.3.1, can be expressed as follows:

$$\begin{aligned} u_{as} &= V_s \sqrt{2}\cos \omega t \\ u_{bs} &= V_s \sqrt{2}\cos(\omega t + \varphi) \end{aligned} \tag{336-337}$$

$$\begin{aligned} u_{as} &= r_s i_{as} + \frac{d\lambda_{as}}{dt} \\ u_{bs} &= r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \\ 0 &= r_r i_{ar} + \frac{d\lambda_{ar}}{dt} \\ 0 &= r_r i_{br} + \frac{d\lambda_{br}}{dt} \end{aligned} \tag{338-342}$$

As the machine is magnetically linear, the fluxes are easily determined from the currents and inductance values. Particularly, it can be written:

$$\lambda_{as} = L_{asas}\,i_{as} + M_{asbs}\,i_{bs} + M_{asar}\,i_{ar} + M_{asbr}\,i_{br}$$
$$\lambda_{bs} = M_{bsas}\,i_{as} + L_{bsbs}\,i_{bs} + M_{bsar}\,i_{ar} + M_{bsbr}\,i_{br}$$
$$\lambda_{as} = M_{aras}\,i_{as} + M_{arbs}\,i_{bs} + L_{arar}\,i_{ar} + M_{arbr}\,i_{br}$$
$$\lambda_{br} = M_{bras}\,i_{as} + M_{brbs}\,i_{bs} + M_{brar}\,i_{ar} + L_{brbr}\,i_{br}$$

(343-346)

The mutual inductances given in the above relations are defined by the subscripts. Applying the reciprocity principle, the following identities are valid:

$$M_{asbs} = M_{bsas} \quad M_{asar} = M_{aras}$$

For further development of the model, the terms from the previously equations are grouped in a matrix form:

$$\lambda_{abs} = \boldsymbol{L}_s \boldsymbol{i}_{abs} + \boldsymbol{L}_{sr} \boldsymbol{i}_{abr}$$
$$\lambda_{abr} = (\boldsymbol{L}_{sr})^{\mathrm{T}} \boldsymbol{i}_{abs} + \boldsymbol{L}_r \boldsymbol{i}_{abr}$$

(347-348)

The indexes denote stator fluxes (as, bs), respectively rotor fluxes (ar, br). The self-inductance can be written by including a leakage inductance caused by the leakage flux and a magnetisation (mutual) inductance caused by the magnetic flux which links the stator and rotor core:

$$L_{ss} = L_{ls} + L_{ms}$$
$$L_{rr} = L_{lr} + L_{mr}$$

(349-350)

where the first term stands for the leakage inductance, and the second one for the mutual inductance between stator and rotor windings. The mutual inductance can be determined:

$$L_{ss(rr)} = \frac{N^2_{s(r)}}{\Re_m}$$

(351)

where: $N_{s(r)}$ is the turns number and $\Re_m$ the magnetic reluctance depending on the air-gap value and on the magnetic core dimensions.

As the magnetic axis of the stator, respectively rotor windings are orthogonal, it results in null mutual inductance between the stator and rotor windings. This is one of the main differences from the three-phase induction machine where the stator (rotor) windings are placed at 120^0 electrical space angle, which determines mutual linkage among the same armature windings.

Due to the relative movement between stator and rotor windings a magnetic linkage will appear. The following defining relations for the mutual inductance can be expressed:

$$M_{asar} = L_{sr}\cos\theta_r$$
$$M_{asbr} = -L_{sr}\sin\theta_r$$
$$M_{bsar} = L_{sr}\sin\theta_r$$
$$L_{bsbs} = L_{sr}\cos\theta_r$$
$$L_{sr} = \frac{N_s N_r}{\Re_m}$$

(352)

The electromagnetic torque of the single-phase induction machine can be determined from the generalised forces law:

$$T_e(i,\theta_r) = \frac{P}{2} \cdot \frac{\partial W_c(i,\theta_r)}{\partial \theta_r}$$

(353)

where W_c represents the conenergy which is equal to the magnetic energy of the linkage field as follows:

$$W_f = \frac{1}{2}\left(L_{ss}i^2_{as} + L_{ss}i^2_{bs} + L'_{rr}\,i'^2_{ar} + L'_{rr}\,i'^2_{br}\right) + L_{ms}i_{as}i'_{ar}\cos\theta_r -$$
$$-L_{ms}i_{as}i'_{br}\sin\theta_r + L_{ms}i_{bs}i'_{ar}\sin\theta_r + L_{ms}i_{bs}i'_{br}\cos\theta_r$$

(354)

In the above relation, the superscript (') stands for referred rotor windings to the stator according to the expressions:

$$i'_{abr} = \frac{N_r}{N_s} i_{abr}$$

$$\lambda'_{abr} = \frac{N_s}{N_r} \lambda_{abr}$$

(356-357)

$$L'_{rr} = \left(\frac{N_s}{N_r}\right)^2 L_{rr}$$

The electromagnetic torque relation can be simplified and presented in the following form:

$$T_e = -\frac{P}{2} L_{ms} \left[\left(i_{as} i'_{ar} + i_{bs} i'_{br} \right) \sin\theta_r + \left(i_{as} i'_{br} - i_{bs} i'_{ar} \right) \cos\theta_r \right]$$

(358)

This relation shows that there are two components of the instantaneous electromagnetic torque: an average component with constant value for a given value of the rotor speed and a pulsating component with a frequency double of the currents frequency (ω_e). The pulsating torque component determines an important magnetic noise for the single-phase induction machine compared to the three-phase induction machine. The mechanical equation that links the torque and the rotor speed is:

$$T_e = J \frac{2}{P} \cdot \frac{d\omega_r}{dt} + B_m \frac{2}{P} \omega_r + T_L$$

(359)

where J is the rotor inertia, B is the viscous friction coefficient associated to the rotational system of the machine and with the mechanical load, and P represents the number of poles for the analysed machine.

3.3. Analysis of the single-phase induction machine in stationary reference frame

In order to eliminate the time dependence of the voltage and flux equations terms, a variables transformation into a new reference frame is necessary. This transformation is given by the following relations (Krause et al - 1995):

$$\boldsymbol{f}_{qds} = \boldsymbol{K}^s_s \boldsymbol{f}_{abs}$$

(360)

$$\boldsymbol{f}'_{qdr} = \boldsymbol{K}^s_r \boldsymbol{f}'_{abr}$$

(361)

$$\boldsymbol{K}^s_s = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

(362)

$$\boldsymbol{K}^s_r = \begin{bmatrix} \cos(\theta - \theta_r) & \sin(\theta - \theta_r) \\ \sin(\theta - \theta_r) & -\cos(\theta - \theta_r) \end{bmatrix}$$

(363)

where f can be fluxes, voltages, currents in new co-ordinates (d-q) or in classical co-ordinates (a-b). Index (s) stands for statoric terms and index (r) stands for the rotoric terms. One should note that the two transformation matrixes depend on the generalised co-ordinate θ which expresses the periferical displacement of the chosen reference frame.

Reference frames linked to stator, rotor or arbitrarily can be chosen for the polyphase induction machine. As for the single-phase induction machine, the stator windings are not identical, the only transformation that maintains the windings parameters (resistance, inductance) unchanged is the stationary reference frame transformation ($\theta = 0$).

The voltage and linkage flux equations are expressed for the previous assumptions made at the beginning of this chapter.

Note: For compactness, the flux linkages will be further described by flux linkage units per second or volts, and the inductance equivalent circuit elements will be transformed in reactance elements.

$$u_{qs} = r_s i_{qs} + \frac{p}{\omega_b} \lambda_{qs}$$

$$u_{dq} = r_s i_{ds} + \frac{p}{\omega_b} \lambda_{ds}$$

$$0 = r'_r i'_{qr} + \frac{p}{\omega_b} \lambda'_{qr} - \frac{\omega_r}{\omega_b} \lambda'_{dr}$$

$$0 = r'_r i'_{dr} + \frac{p}{\omega_b} \lambda'_{dr} + \frac{\omega_r}{\omega_b} \lambda'_{qr}$$

(364-367)

$$\lambda_{qs} = x_s i_{qs} + x_m (i_{qs} + i'_{qr})$$

$$\lambda_{ds} = x_s i_{ds} + x_m (i_{ds} + i'_{dr})$$

$$\lambda'_{qr} = x'_r i'_{qr} + x_m (i_{qs} + i'_{qr})$$

$$\lambda'_{dr} = x'_r i'_{dr} + x_m (i_{ds} + i'_{dr})$$

(368-371)

$$T_e = \frac{P}{2} \cdot \frac{x_m}{\omega_b} \cdot (i_{qs} i'_{dr} - i_{ds} i'_{qr})$$

(372)

In the reference frame systems theory we have to note that the parameters do not depend on the relative position between the stator and the rotor. The time derivative (*d/dt*) in the above equations is denoted as *p,* and ω_b and ω_r represent the angular base speed given by the supply frequency, respectively the electrical rotor speed. The *qr* terms are referred to the *qs* winding and the *dr* terms are referred to the *ds* winding. It is important to observe that stator windings of the single-phase induction machine physically represent the *d-q* co-ordinates windings, different from the three-phase induction machine where the two-axis co-ordinate equivalent windings are fictive.

The dynamic analysis of the symmetrical single-phase induction machine can be accomplished by using the equivalent circuit from Fig. 3.2. The only modification that has to be done in order to obtain the real value of the terminal quantities (voltage, current) is the multiplication by (-1) factor for the auxiliary (*ds*) winding parameters.
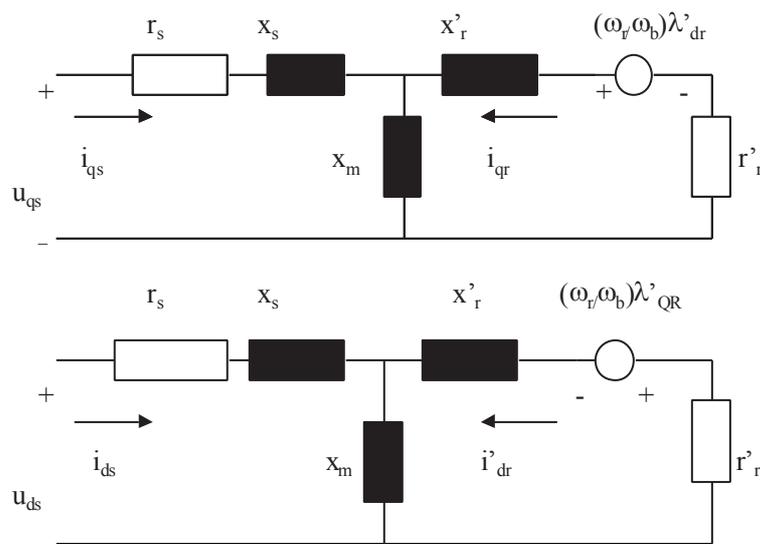
Fig. 3.2. Equivalent circuit of the symmetrical single-phase induction machine conform to reference frame system theory

3.4. Analysis of the steady-state operation for the symmetrical single-phase induction machine

In a wide range of applications the single-phase induction machine is equipped with a cage rotor. During the steady-state operation, the stator parameters are variable with the stator voltage frequency ω, and the rotor parameters are variable with the slip frequency $\omega - \omega_r$. From the stationary reference frame equations, by letting the differential operator p be replaced by the complex operator $j\omega$, the following matrix relation is established (Krause and Thomas - 1965):

$$\begin{bmatrix} \tilde{U}_{qs} \\ \tilde{U}_{ds} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_s + j\dfrac{\omega}{\omega_b}x_{ss} & 0 & j\dfrac{\omega}{\omega_b}x_m & 0 \\ 0 & r_s + j\dfrac{\omega}{\omega_b}x_{ss} & 0 & j\dfrac{\omega}{\omega_b}x_m \\ j\dfrac{\omega}{\omega_b}x_m & -\dfrac{\omega_r}{\omega_b}x_m & r'_r + j\dfrac{\omega}{\omega_b}x'_{rr} & -\dfrac{\omega_r}{\omega_b}x'_{rr} \\ \dfrac{\omega_r}{\omega_b}x_m & j\dfrac{\omega_r}{\omega_b}x_m & \dfrac{\omega_r}{\omega_b}x'_{rr} & r'_r + j\dfrac{\omega}{\omega_b}x'_{rr} \end{bmatrix} \cdot \begin{bmatrix} \tilde{I}_{qs} \\ \tilde{I}_{ds} \\ \tilde{I}_{qr} \\ \tilde{I}_{dr} \end{bmatrix} \tag{373}$$

where

$$x_{ss} = x_s + x_m$$
$$x'_{rr} = x'_r + x_m \tag{374-375}$$

A symmetrical two-phase system is defined by the identities:

$$\tilde{F}_{QS} = -j\tilde{F}_{DS}$$
$$\tilde{F}_{QR} = -j\tilde{F}_{DR} \tag{376-377}$$

where $\tilde{F}$ represents a complex variable with current or voltage significance. The four equations from the above matrix relation are interdependent. If it is used, the variable slip s is defined as:

$$s = \frac{\omega - \omega_r}{\omega} \tag{378}$$

and also through the inverse transformation from the stationary reference frame (*d-q*) to the physical one (*a-b*), the relations for the steady-state operation analysis of the symmetrical single-phase induction machine can be written:

$$\tilde{U}_{as} = (r_s + jx_s)\tilde{I}_{as} + jx_m(\tilde{I}_{as} + \tilde{I}_{ar})$$
$$0 = \left(\frac{r'_r}{s} + jx'_r\right)\tilde{I}_{ar} + jx_m(\tilde{I}_{as} + \tilde{I}_{ar}) \tag{379-380}$$

The above equations suggest the equivalent circuit from Fig. 3.3.
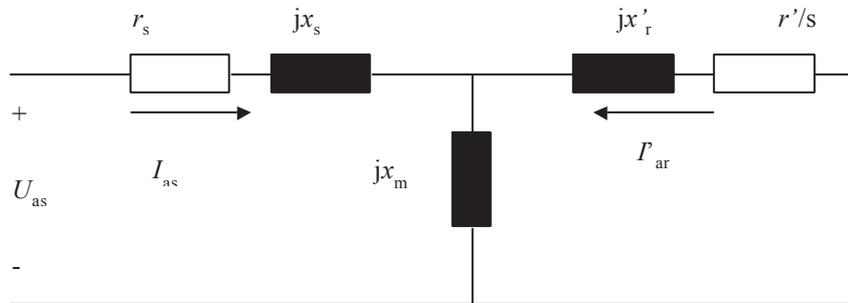
Fig. 3.3. Equivalent circuit of the symmetrical single-phase induction machine for the steady-state operation analysis

With slight modifications (x_m becomes $3/2x_m$) it can be observed that the equivalent circuit from Fig. 3.3 can be used also for the three-phase induction machine performance in steady-state operation.

The electromagnetic torque equation is:

$$T_e = 2\left(\frac{P}{2}\right)\frac{x_m}{\omega_b}\text{Re}\left[j\tilde{I}_{as}^*\tilde{I}_{ar}\right] = \frac{2(P/2)(x_m^2/\omega_b)(r'_r/s)\left|\tilde{I}_{as}\right|^2}{(r'_r/s)^2 + x'^2_{rr}} \tag{381}$$

$$T_e = \frac{2(P/2)(x_m^2/\omega_b)(r'_r s)\left|\tilde{U}_{as}\right|^2}{(r_s r'_r + s(x_m^2 - x_{ss}x'_{rr}))^2 + (r_r x_{ss} + sr_s x'_{rr})^2} \tag{382}$$

It is important to highlight that the positive values of the torque are obtainable when the slip s is positive (*motor* operation) and the negative values when the slip s is negative (*generator* operation). By setting the torque/slip derivative equal to zero, the relation for the critical slip can be obtained:

$$s_m = r'_r G$$

$$G = \pm\sqrt{\frac{r_s^2 + x_{ss}^2}{(x_m^2 - x_{ss}x'_{rr})^2 + r_s^2 x'^2_{rr}}} \tag{383-384}$$

The positive value corresponds to *motor* operation, and the negative one to *generator* operation. If at start-up ($s = 1$) the torque is directly dependent to the variation of rotor resistance, as the magnetisation reactance value is considered to be much higher than the stator or rotor resistance value, the maximum torque value is not dependent to the rotor resistance value:

$$T_{e,max} = \frac{2(P/2)(x_m^2/\omega)G\left|\tilde{U}_{as}\right|^2}{\left[r_s + G(x_m^2 - x_{ss}x'_{rr})\right]^2 + (x_{ss} + Gr_s x'_{rr})^2} \tag{385}$$

3.5. Analysis of the unsymmetrical single-phase induction machine

A new equations set and a new equivalent circuit for the unsymmetrical single-phase induction machine can be obtained by eliminating from the initial assumptions the one referring to identical stator windings (Krause - 1965):

Voltage equations:

$$u_{qs} = r_m i_{qs} + \frac{p}{\omega_b}\lambda_{qs}$$

$$u_{ds} = r_a i_{ds} + \frac{p}{\omega_b}\lambda_{ds}$$

$$0 = r'_r i'_{qr} + \frac{p}{\omega_b}\lambda'_{qr} - \frac{1}{k}\cdot\frac{\omega_r}{\omega_b}\lambda'_{qr} \tag{386-389}$$

$$0 = k^2 r'_r i'_{dr} + \frac{p}{\omega_b}\lambda'_{dr} + k\cdot\frac{\omega_r}{\omega_b}\lambda'_{dr}$$

Flux equations:

$$\lambda_{qs} = x_{lm}i_{qs} + x_m(i_{qs} + i'_{qr})$$

$$\lambda_{ds} = x_{la}i_{ds} + k^2 x_m(i_{ds} + i'_{dr})$$

$$\lambda'_{qr} = x'_{sr}i'_{qr} + x_m(i_{qs} + i'_{qr}) \tag{390-393}$$

$$\lambda'_{dr} = k^2 x'_{sr}i'_{dr} + k^2 x_m(i_{ds} + i'_{dr})$$

The electromagnetic torque equation is:

$$T_e = \frac{P}{2} \cdot \frac{x_m}{\omega_b} \cdot k \cdot (i_{qs} i'_{dr} - i_{ds} i'_{qr}) \tag{394}$$

The mathematical model presented in Fig. 3.4 permits the analysis of steady state as well as dynamic operation of the unsymmetrical single-phase induction machine. This model is also suitable for implementing non-linearities effects such as: core loss or saturation of the main or leakage inductance.
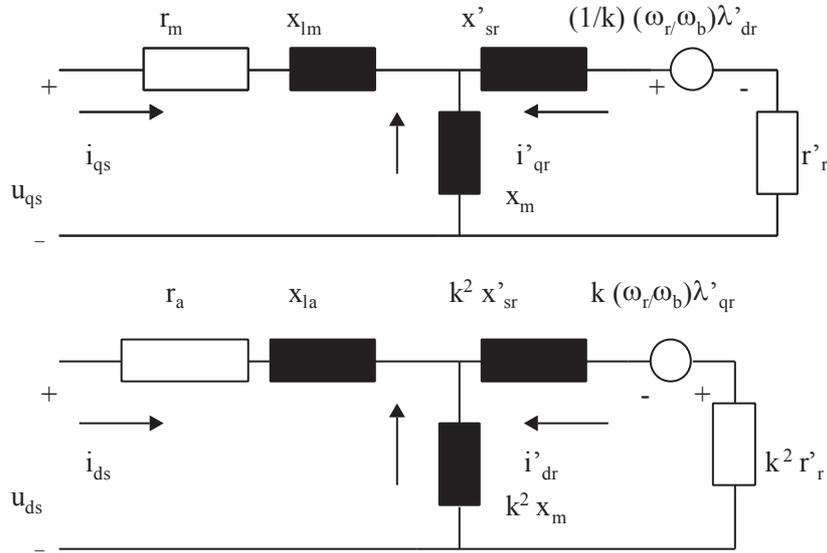
Fig. 3.4. Equivalent circuit of the unsymmetrical single-phase induction machine

3.6. Linear models for single-phase induction motors

The analysis of the induction machine-drives is generally made using a conventional linear mathematical machine model, either in the form of self and mutual inductances, or the familiar T-form of equivalent circuit. This type of model has been developed in the previous paragraph, dedicated to symmetrical and single-phase induction machines.

When the vector control of these machines is implemented, such kind of models is adequate for many situations, especially for air-gap flux orientation control strategy. However, they are more complex than necessary for the analysis of most linear machines.

If for the polyphased induction machine two approaches are valid related to vector control analysis, i.e. the space vector notation and the two-axis co-ordinates reference frame, the single-phase induction machine can be completely described only by the latter approach in stationary reference frame. As for the polyphased induction machines, the single-phase version can be controlled with a scheme which maintains correct angular relationship between the stator current vector and one flux vector (stator, air-gap or rotor flux), by either direct or indirect methods. All the field-oriented methods suffer from specific theoretical and practical problems. The indirect methods are highly dependent on the machine rotor parameters (varying with load and temperature), and have good speed control performance only if precise shat encoders are used to calculate the electrical frequency. The direct methods present minor dependency on rotor parameters, but are not able to measure the selected flux vector at low speed, i.e. zero frequency. This is why only estimations of the selected flux vector can make possible the total control of the speed in direct methods. For estimation, usually the terminal quantities are measured and a mathematical model with related parameters is used. Thus, the correct estimation of the machine parameters is essential for all types of field oriented control schemes.

The original field orientation scheme, developed for the three-phase induction machine, was based on the alignment of the rotor flux linkage because the torque and the rotor flux are related to each other in a straightforward manner, without any de-coupling circuit. To calculate the rotor flux, the stator and rotor leakage inductance and the main inductance are necessary. If some other equivalent circuits are implemented, it can be obtained a Γ model like that one illustrated by Slemon (1989), where the stator and rotor leakage inductance are viewed as a total measurable inductance.

3.6.1. Linear Γ model of the symmetrical single-phase induction machine

The classical T-form circuit model can be transformed into simpler models with no loss of information or accuracy. Since rotor variables can be seen from the stator reference frame as referred variables depending on the transformation turn ratio, we can choose a value such that the magnetisation inductance is equal to the total stator inductance. This would give the following set of rotor variables related to those from the T-form equivalent circuit:

$$
\begin{bmatrix} \lambda'_{dR} \\ \lambda'_{qR} \\ i'_{dR} \\ i'_{qR} \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \\ \dfrac{1}{\gamma} \\ \dfrac{1}{\gamma} \end{bmatrix} \cdot \begin{bmatrix} \lambda'_{dr} & \lambda'_{qr} & i'_{dr} & i'_{qr} \end{bmatrix}
\tag{395}
$$

where $\gamma = x_m /(x_s + x_m)$. The above relation combined with the voltage equations gives the equivalent d-q modified model for the symmetrical single-phase induction machine presented in Fig. 3.5. This configuration has been denoted as the Γ form model due to its inductance structure. The parameters of this equivalent circuit are related to those of the T-form d-q mathematical model through the relations:

$$
\begin{aligned}
x_M &= \gamma \cdot x_m = x_s + x_m \\
x_L &= (x_r + x_m) \cdot \gamma^2 - (x_s + x_m) = x_s \cdot \gamma + x_r \cdot \gamma^2 \\
r'_R &= \gamma^2 \cdot r'_r
\end{aligned}
\tag{396}
$$
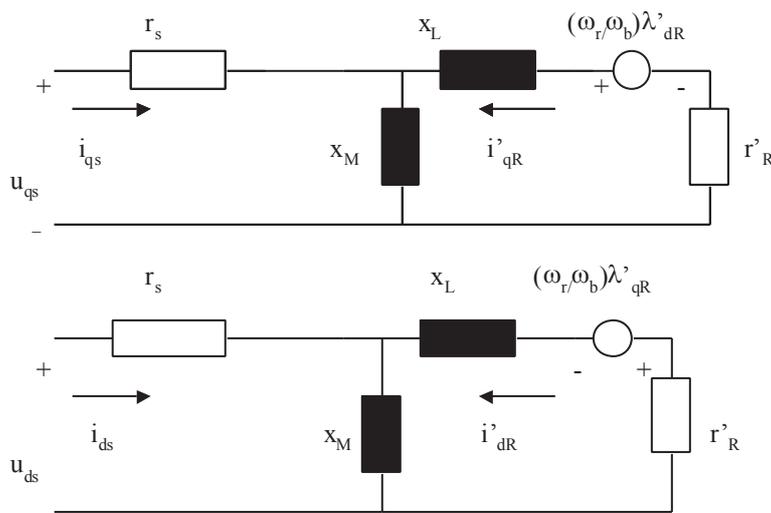
Fig. 3.5. Equivalent linear *d-q* Γ form circuit for the symmetrical single-phase induction machine

These parameters can be derived directly from the usual no load and standstill measurements on the machine.

This Γ model represents an appropriate solution for the analysis of scalar control and vector control (stator flux oriented) single-phase induction machine (Slemon - 1994). The following equations describe the complete operation of the system:

Stator and rotor voltage:

$$u_{qs} = r_s i_{qs} + \frac{p}{\omega_b} \lambda_{qs}$$

$$u_{ds} = r_s i_{ds} + \frac{p}{\omega_b} \lambda_{ds}$$

$$0 = r'_R i'_{qR} + \frac{p}{\omega_b} \lambda'_{qR} - \frac{\omega_r}{\omega_b} \lambda'_{dR}$$

(397-400)

$$0 = r'_R i'_{dR} + \frac{p}{\omega_b} \lambda'_{dR} + \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

Stator and rotor flux linkage:

$$\lambda_{qs} = x_M i_{qM} = x_M (i_{qs} + i'_{qR})$$

$$\lambda_{ds} = x_M i_{dM} = x_M (i_{ds} + i'_{dR})$$

$$\lambda'_{qR} = x_L i'_{qR} + x_M i_{qM} = x_M i_{qs} + (x_L + x_M) i'_{qR}$$

(401-404)

$$\lambda'_{dR} = x_L i'_{dR} + x_M i_{dM} = x_M i_{ds} + (x_L + x_M) i'_{dR}$$

Electromagnetic torque:

$$T_e = \frac{P}{2} \cdot \frac{x_M}{\omega_b} \cdot \left(i_{qs} i'_{dR} - i_{ds} i'_{qR} \right) = \frac{P}{2} \cdot \left(i_{qs} \lambda_{ds} - i_{ds} \lambda_{qs} \right)$$

(405)

In the steady-state ($p = 0$) the torque expression becomes:

$$T_e = 2 \left(\frac{P}{2} \right) \cdot \frac{\left(\lambda_{qs}^2 + \lambda_{ds}^2 \right) r'_R s \omega_b}{r'^2_R + \left(s x_L \right)^2}$$

(406)

where the slip s is given by: $s = (\omega_b - \omega_r) / \omega_b$

3.6.2. Linear inverse Γ model of the symmetrical single-phase induction machine

If the arbitrarily turns ratio of referring the rotor parameters to the stator is chosen such that the magnetisation inductance is equal to the total rotor inductance, a new set of variables is obtainable:

$$\begin{bmatrix} \lambda''_{dR} \\ \lambda''_{qR} \\ i''_{dR} \\ i''_{qR} \end{bmatrix} = \begin{bmatrix} \gamma' \\ \gamma' \\ \dfrac{1}{\gamma'} \\ \dfrac{1}{\gamma'} \end{bmatrix} \cdot \begin{bmatrix} \lambda'_{dr} & \lambda'_{qr} & i'_{dr} & i'_{qr} \end{bmatrix}$$

(407)

where: $\gamma'' = x_m / (x_m + x_r)$. Using the voltage equations given for T-form equivalent circuit, it will result a new configuration denoted as inverse Γ form model shown in Fig.3.6 in which:

$$x'_L = x_s + \gamma' \cdot x_r = \frac{(x_s + x_m)(x_r + x_m) - x_m^2}{(x_r + x_m)}$$

$$x'_M = \gamma' \cdot x_m = \gamma'^2 \cdot (x_r + x_m) \tag{408}$$

$$r''_R = \gamma'^2 \cdot r'_r$$

The inverse Γ form mathematical model of the single-phase induction machine is particularly appropriate to analyse the vector-controlled machine in rotor field oriented systems.
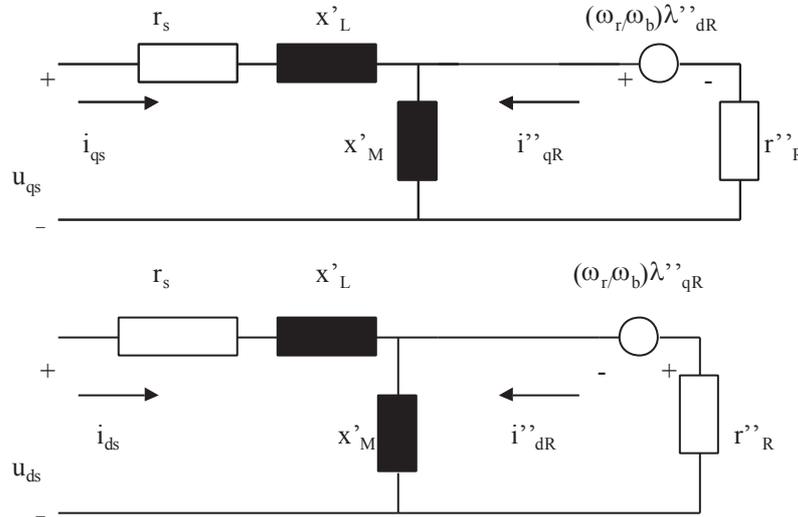
Fig. 3.6. Equivalent linear d-q inverse Γ form circuit for the symmetrical single-phase induction machine

The following equations can be written to describe the operation of the machine:
Stator and rotor voltage:

$$u_{qs} = r_s i_{qs} + \frac{p}{\omega_b} \lambda_{qs}$$

$$u_{ds} = r_s i_{ds} + \frac{p}{\omega_b} \lambda_{ds}$$

$$0 = r''_R i''_{qR} + \frac{p}{\omega_b} \lambda''_{qR} - \frac{\omega_r}{\omega_b} \lambda''_{dR} \tag{409-412}$$

$$0 = r''_R i''_{dR} + \frac{p}{\omega_b} \lambda''_{dR} + \frac{\omega_r}{\omega_b} \lambda''_{qR}$$

Stator and rotor flux linkage:

$$\lambda_{qs} = x'_L i_{qs} + x'_M (i_{qs} + i''_{qR}) = (x'_M + x'_L) i_{qs} + x'_M i'_{qR}$$

$$\lambda_{ds} = x'_L i_{ds} + x'_M (i_{ds} + i''_{dR}) = (x'_M + x'_L) i_{ds} + x'_M i'_{dR}$$

$$\lambda''_{qR} = x'_M i'_{qM} = x'_M (i_{qs} + i''_{qR}) \tag{413-416}$$

$$\lambda''_{dR} = x'_M i'_{dM} = x'_M (i_{ds} + i''_{dR})$$

Electromagnetic torque:

$$T_e = \left(\frac{P}{2}\right) \cdot \left(\frac{x'_M}{\omega_b}\right)(i_{qs} i''_{dR} - i_{ds} i''_{qR}) = \left(\frac{P}{2}\right) \cdot (i_{qs} \lambda''_{dR} - i_{ds} \lambda''_{qR}) = \left(\frac{P}{2}\right) \cdot (i''_{dR} \lambda''_{qR} - i''_{qR} \lambda''_{dR}) \tag{417}$$

In steady-state operation the torque becomes:

$$T_{\mathrm{e}} = 2 \cdot \left(\frac{P}{2} \right) \cdot \frac{\left(\lambda''^2_{\mathrm{dR}} + \lambda''^2_{\mathrm{qR}} \right) \cdot s\omega_{\mathrm{b}}}{r''_{\mathrm{R}}} \qquad (418)$$

where s denotes the slip for the analysed machine.

3.6.3. Universal model of the symmetrical single-phase induction machine

The modern theory of the vector controlled induction machine demonstrates that several reference flux vectors can be chosen for an independent control of the torque and flux in the machine (DeDoncker et al - 1995). A unified theory simplifies and unifies the calculations of the machine parameters and makes a change of the reference vector flux more flexible. The idea is to link the stationary reference frame (the only valid option for single-phase induction machine) to an arbitrary flux vector $\underline{\lambda}_\gamma$ which can be derived from the flux linkage equations given at the previous analysed models (T-form, Γ-form, inverse Γ-form) by means of a turn ratio transformation.

An arbitrary turn ratio γ is used to multiply the real rotor vector flux $\underline{\lambda}_{\mathrm{r}} = \lambda_{\mathrm{dr}} + j \cdot \lambda_{\mathrm{qr}}$ and defining the flux linkage equations as:

$$\lambda_{\mathrm{ds}} = \left(x_{\mathrm{s}} + x_{\mathrm{m}} \right) \cdot i_{\mathrm{ds}} + x_{\mathrm{m}} i_{\mathrm{dr}} - \gamma \cdot x_{\mathrm{m}} i_{\mathrm{ds}} + \gamma \cdot x_{\mathrm{m}} i_{\mathrm{ds}}$$
$$\lambda_{\mathrm{qs}} = \left(x_{\mathrm{s}} + x_{\mathrm{m}} \right) \cdot i_{\mathrm{qs}} + x_{\mathrm{m}} i_{\mathrm{qr}} - \gamma \cdot x_{\mathrm{m}} i_{\mathrm{qs}} + \gamma \cdot x_{\mathrm{m}} i_{\mathrm{qs}} \qquad (419\text{-}420)$$

$$\lambda'_{\mathrm{dR}} = \gamma \cdot \lambda'_{\mathrm{dr}} = \gamma \cdot x_{\mathrm{m}} i_{\mathrm{ds}} + \gamma \cdot \left(x_{\mathrm{r}} + x_{\mathrm{m}} \right) \cdot i_{\mathrm{dr}} - x_{\mathrm{m}} i_{\mathrm{dr}} + x_{\mathrm{m}} i_{\mathrm{dr}}$$
$$\lambda'_{\mathrm{qR}} = \gamma \cdot \lambda'_{\mathrm{qr}} = \gamma \cdot x_{\mathrm{m}} i_{\mathrm{qs}} + \gamma \cdot \left(x_{\mathrm{r}} + x_{\mathrm{m}} \right) \cdot i_{\mathrm{qr}} - x_{\mathrm{m}} i_{\mathrm{qr}} + x_{\mathrm{m}} i_{\mathrm{qr}} \qquad (421\text{-}423)$$

The transformed rotor current $i'_{\mathrm{d(q)R}}$ equals:

$$i'_{\mathrm{dR}} = \frac{i'_{\mathrm{dr}}}{\gamma}; \ \ i'_{\mathrm{qR}} = \frac{i'_{\mathrm{qr}}}{\gamma} \qquad (423)$$

The flux linkage equations reduce to:

$$\lambda_{\mathrm{ds}} = \left(x_{\mathrm{s}} + \left(1 - \gamma \right) \cdot x_{\mathrm{m}} \right) \cdot i_{\mathrm{ds}} + \gamma \cdot x_{\mathrm{m}} \cdot \left(i_{\mathrm{ds}} + i'_{\mathrm{dR}} \right) = \left(x_{\mathrm{s}} + \left(1 - \gamma \right) \cdot x_{\mathrm{m}} \right) \cdot i_{\mathrm{ds}} + \lambda'_{\mathrm{d\gamma}}$$

$$\lambda_{\mathrm{qs}} = \left(x_{\mathrm{s}} + \left(1 - \gamma \right) \cdot x_{\mathrm{m}} \right) \cdot i_{\mathrm{qs}} + \gamma \cdot x_{\mathrm{m}} \cdot \left(i_{\mathrm{qs}} + i'_{\mathrm{qR}} \right) = \left(x_{\mathrm{s}} + \left(1 - \gamma \right) \cdot x_{\mathrm{m}} \right) \cdot i_{\mathrm{qs}} + \lambda'_{\mathrm{q\gamma}}$$

$$\lambda'_{\mathrm{dR}} = \gamma \cdot \left(\gamma \cdot x_{\mathrm{r}} + \left(\gamma - 1 \right) \cdot x_{\mathrm{m}} \right) \cdot i'_{\mathrm{dR}} + \gamma \cdot x_{\mathrm{m}} \cdot \left(i_{\mathrm{ds}} + i'_{\mathrm{dR}} \right) = \lambda'_{\mathrm{d\gamma}} + \gamma \cdot \left(\gamma \cdot x_{\mathrm{r}} + \left(\gamma - 1 \right) \cdot x_{\mathrm{m}} \right) \cdot i'_{\mathrm{dR}}$$

$$\lambda'_{\mathrm{qR}} = \gamma \cdot \left(\gamma \cdot x_{\mathrm{r}} + \left(\gamma - 1 \right) \cdot x_{\mathrm{m}} \right) \cdot i'_{\mathrm{qR}} + \gamma \cdot x_{\mathrm{m}} \cdot \left(i_{\mathrm{qs}} + i'_{\mathrm{qR}} \right) = \lambda'_{\mathrm{q\gamma}} + \gamma \cdot \left(\gamma \cdot x_{\mathrm{r}} + \left(\gamma - 1 \right) \cdot x_{\mathrm{m}} \right) \cdot i'_{\mathrm{qR}}$$

$$(424\text{-}427)$$

This flux linkage can be represented by the equivalent circuit given in Fig. 3.7.
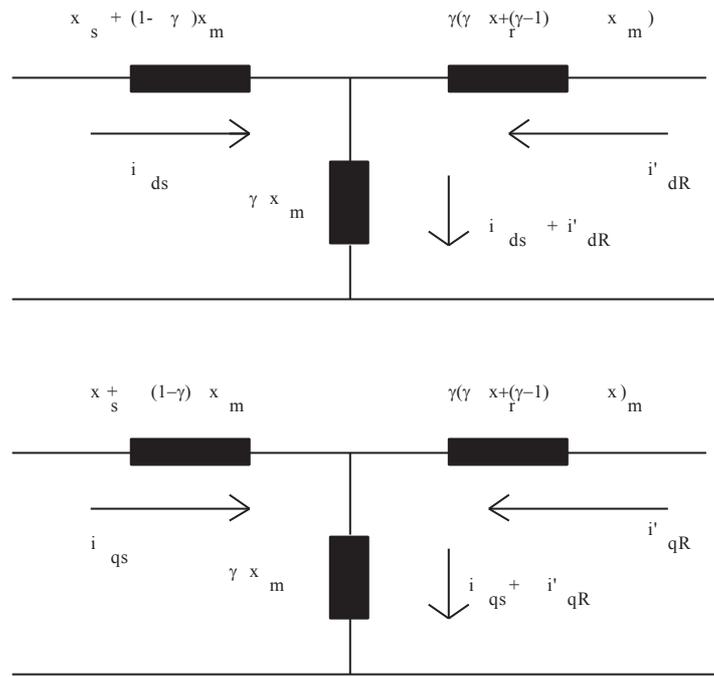
Fig. 3.7. Flux linkage equivalent circuit with arbitrary turn ratio

The rotor quantities are now calculated with the equations:

$$0 = r'_R i'_{dR} + \frac{p}{\omega_b} \lambda'_{dR} + \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

$$0 = r'_R i'_{qR} + \frac{p}{\omega_b} \lambda'_{qR} - \frac{\omega_r}{\omega_b} \lambda'_{dR} \qquad (428\text{-}430)$$

$$r'_R = \gamma^2 \cdot r_r$$

Through variation of the arbitrary turn ratio "γ" the appropriate reference vector flux for controlling the machine can be selected. Table 3.I summarise the choices that have to be made:

TABLE 3.I . Specific choices of the turn ratio "γ"

Turn ratio	Mathematical model	$\lambda_{d(q)\gamma}$	Flux selected
$\gamma = \dfrac{x_m}{x_m + x_r}$	Inverse Γ-form	$\lambda_{d(q)r}$	Rotor flux
$\gamma = 1$	T-form	$\lambda_{d(q)m}$	Air-gap flux
$\gamma = \dfrac{x_m + x_s}{x_m}$	Γ-form	$\lambda_{d(q)s}$	Stator flux

The universal mathematical model given in Fig. 3.7 permits a choice between different flux vectors by selecting only one parameter "γ" , denoted the arbitrary turn ratio. All other basic machine parameters remain unaffected.

3.6.4. Linear Γ model of the unsymmetrical single-phase induction machine

Usually the single-phase induction machine is equipped with non-identical sinusoidally distributed windings arranged in space quadrature. From the T-form equivalent circuit developed

for this type of machine in previous paragraphs, it can be developed a linear Γ-form equivalent circuit as shown in Fig. 3.8.

Since rotor variables can be seen from the stator reference frame as referred variables depending on the transformation turn ratio, we can choose a value such that the magnetisation inductance is equal to the total stator inductance on each of the two-axis co-ordinates. This would give the following set of rotor variables related to those from the T-form equivalent circuit:

$$\begin{bmatrix} \lambda'_{dR} \\ \lambda'_{qR} \\ i'_{dR} \\ i'_{qR} \end{bmatrix} = \begin{bmatrix} \gamma_d \\ \gamma_q \\ \dfrac{1}{\gamma_d} \\ \dfrac{1}{\gamma_q} \end{bmatrix} \cdot \begin{bmatrix} \lambda'_{dr} & \lambda'_{qr} & i'_{dr} & i'_{qr} \end{bmatrix} \tag{431}$$

where

$$\gamma_q = x_m /(x_{lm} + x_m)$$
$$\gamma_d = k^2 x_m /(x_{la} + k^2 x_m) \tag{432-433}$$

For an identical distribution of the windings, we have with a good approximation $x_{la} = k^2 x_{lm}$, and this gives equal turn ratio for each axis ($\gamma_d = \gamma_q$).

The above relations combined with the voltage equations give the equivalent d-q modified model for the unsymmetrical single-phase induction machine. The parameters of the Γ-form model are related to those of the T-form d-q mathematical model through the relations:

$$x_{Mq} = \gamma_q \cdot x_m = x_{lm} + x_m$$
$$x_{Lq} = (x_r + x_m) \cdot \gamma_q^2 - (x_{lm} + x_m) = x_{lm} \cdot \gamma_q + x_r \cdot \gamma_q^2$$
$$r'_R = \gamma_q^2 \cdot r'_r \tag{434-438}$$
$$x_{Md} = \gamma_d \cdot k^2 x_m = x_{sa} + k^2 x_m \cong k^2 x_{Mq}$$
$$x_{Ld} = k^2 (x_r + x_m) \cdot \gamma_d^2 - (x_{la} + k^2 x_m) = x_{la} \cdot \gamma_d + k^2 x_r \cdot \gamma_q^2 \cong k^2 x_{Lq}$$
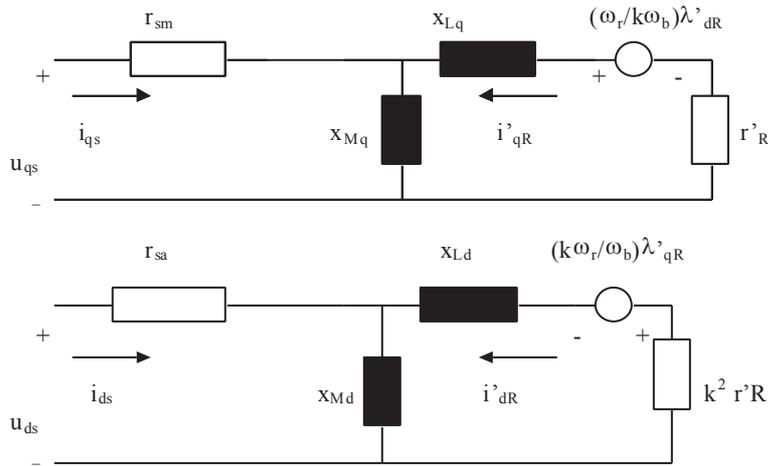
Fig. 3.8. Equivalent linear d-q Γ form circuit for the unsymmetrical single-phase induction machine

It results an equations system as follows:
Stator and rotor voltage:

$$u_{qs} = r_{sm} i_{qs} + \frac{p}{\omega_b} \lambda_{qs}$$

$$u_{ds} = r_{sa} i_{ds} + \frac{p}{\omega_b} \lambda_{ds}$$

$$0 = r'_R i'_{qR} + \frac{p}{\omega_b} \lambda'_{qR} - \frac{1}{k} \cdot \frac{\omega_r}{\omega_b} \lambda'_{dR}$$ (439-442)

$$0 = k^2 r'_R i'_{dR} + \frac{p}{\omega_b} \lambda'_{dR} + k \cdot \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

Flux linkage:

$$\lambda_{qs} = x_{Mq} \cdot \left(i_{qs} + i'_{qR} \right)$$

$$\lambda_{ds} = x_{Md} \cdot \left(i_{ds} + i'_{dR} \right) = k^2 x_{Mq} \cdot \left(i_{ds} + i'_{dR} \right)$$

$$\lambda'_{qR} = x_{Mq} \cdot i_{qs} + \left(x_{Mq} + x_{Lq} \right) \cdot i'_{qR}$$ (443-447)

$$\lambda'_{dR} = x_{Md} \cdot i_{ds} + \left(x_{Md} + x_{Ld} \right) \cdot i'_{dR} = k^2 \cdot \left[x_{Mq} \cdot i_{ds} + \left(x_{Mq} + x_{Lq} \right) \cdot i'_{dR} \right]$$

Electromagnetic torque:

$$T_e = \frac{P}{2} \cdot k \cdot \frac{x_{Mq}}{\omega_b} \left(i_{qs} i'_{dR} - i_{ds} i'_{qR} \right) = \frac{P}{2} \cdot \left(\frac{1}{k} \cdot i_{qs} \lambda_{ds} - k \cdot i_{ds} \lambda_{qs} \right)$$ (448)

This Γ-form model represents an appropriate solution for the analysis of scalar control and vector control (stator flux oriented) of the unsymmetrical single-phase induction machine drives.

3.6.5. Linear inverse Γ model of the unsymmetrical single-phase induction machine

Another possible approach on analysing the unsymmetrical single-phase induction machine is the one when an arbitrary turn ratio is used such that the magnetisation inductance in each axis is equal to the correspondent total rotor inductance. The following set of variables is readily obtainable:

$$\begin{bmatrix} \lambda''_{dR} \\ \lambda''_{qR} \\ i''_{dR} \\ i''_{qR} \end{bmatrix} = \begin{bmatrix} \gamma_d{'} \\ \gamma_q{'} \\ \dfrac{1}{\gamma_d{'}} \\ \dfrac{1}{\gamma_q{'}} \end{bmatrix} \cdot \begin{bmatrix} \lambda'_{dr} & \lambda'_{qr} & i'_{dr} & i'_{qr} \end{bmatrix}$$ (449)

where:

$$\gamma_q{'} = \frac{x_m}{x_r + x_m} \qquad \gamma_d{'} = \frac{k^2 x_m}{k^2 (x_r + x_m)} = \gamma_q{'}$$ (450)

The linear inverse Γ-form model of the unsymmetrical single-phase induction machine is presented in Fig. 3.9. This configuration is deductible from the T-form model using the above expressions and new equivalent machine parameters defined as follows:

$$x'_{Mq} = \gamma_q{}' \cdot x_m$$

$$x'_{Md} = \gamma_d{}' \cdot k^2 x_m = k^2 x'_{Mq}$$

$$x'_{Lq} = x_{lm} + \gamma_q{}' \cdot x_r \qquad (451\text{-}455)$$

$$x'_{Ld} = x_{la} + \gamma_d{}' \cdot k^2 x_r \cong k^2 x'_{Lq}$$

$$r'_R = \left(\gamma'_{q(d)}\right)^2 \cdot r'_r$$
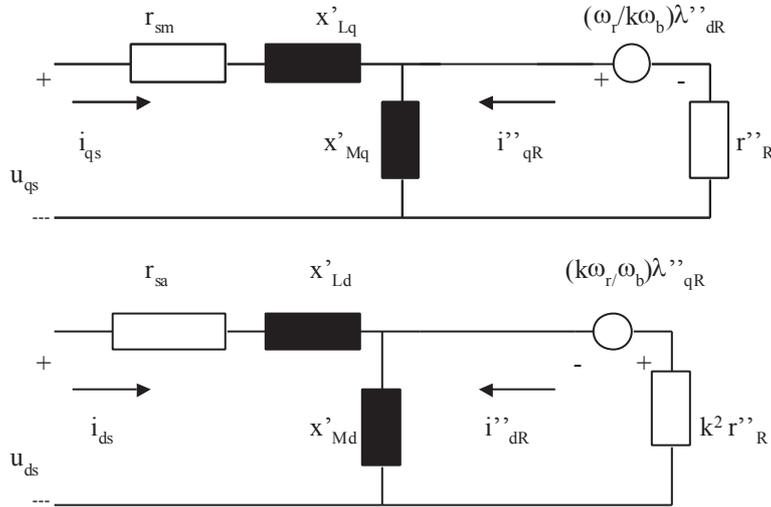
Fig.3.9. Equivalent linear *d-q* inverse Γ form circuit for the unsymmetrical single-phase induction machine

The complete equations system is detailed below:

Stator and rotor voltage:

$$u_{qs} = r_{sm} i_{qs} + \frac{p}{\omega_b} \lambda_{qs}$$

$$u_{ds} = r_{sa} i_{ds} + \frac{p}{\omega_b} \lambda_{ds}$$

$$0 = r'_R i'_{qR} + \frac{p}{\omega_b} \lambda'_{qR} - \frac{1}{k} \cdot \frac{\omega_r}{\omega_b} \lambda'_{dR} \qquad (456\text{-}459)$$

$$0 = k^2 r'_R i'_{dR} + \frac{p}{\omega_b} \lambda'_{dR} + k \cdot \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

Flux linkage:

$$\lambda_{qs} = x'_{Lq} i_{qs} + x'_{Mq}(i_{qs} + i'_{qR}) = (x'_{Mq} + x'_{Lq})i_{qs} + x'_{Mq} i'_{qR}$$

$$\lambda_{ds} = x'_{Ld} i_{ds} + x'_{Md}(i_{ds} + i'_{dR}) = (x'_{Md} + x'_{Ld})i_{ds} + x'_{Md} i'_{dR} =$$

$$= k^2 \cdot \left((x'_{Mq} + x'_{Lq})i_{ds} + x'_{Mq} i'_{dR}\right) \qquad (460\text{-}463)$$

$$\lambda'_{qR} = x'_{Mq} i'_{qM} = x'_{Mq}(i_{qs} + i'_{qR})$$

$$\lambda'_{dR} = x'_{Md} i'_{dM} = x'_{Md}(i_{ds} + i'_{dR}) = k^2 x'_{Mq}(i_{ds} + i'_{dR})$$

Electromagnetic torque:

$$T_e = \left(\frac{P}{2}\right) \cdot k \cdot \frac{x'_{Mq}}{\omega_b} \cdot \left(i_{qs} i'_{dR} - i_{ds} i'_{qR}\right) = \left(\frac{P}{2}\right) \cdot \left(\frac{1}{k} \cdot i_{qs} \lambda'_{dR} - k \cdot i_{ds} \lambda'_{qR}\right) =$$

$$= \left(\frac{P}{2}\right) \cdot \left(k \cdot i'_{dR} \lambda'_{qR} - \frac{1}{k} \cdot i'_{qR} \lambda'_{dR}\right)$$

(464)

The circuit model from Fig. 3.9, is particularly suitable for analysis and understanding the vector control systems with single-phase induction machine when the rotor flux vector is chosen as reference.

3.6.6. Universal model of the unsymmetrical single-phase induction machine

All the developed mathematical models for the unsymmetrical induction machine can be generically presented with a universal model that preserves the characteristics for each of the equivalent form circuits.

The basic idea is to calculate in the stationary reference frame (the only valid option for single-phase induction machine that maintains constant parameters) an arbitrary flux vector. The flux vector is determined from the flux linkage equations given in various forms of equivalent circuits (T-form, Γ-form, inverse Γ-form) by using an arbitrary turn ratio transformation.

This turn ratio, that characterises the two-axis reference frame, has two values: γ_q and γ_d. By multiplying the real rotor vector flux $\underline{\lambda}_r = \lambda_{dr} + j \cdot \lambda_{qr}$ with the turn ratio we can define the flux linkage equations as:

$$\lambda_{qs} = \left(x_{lm} + x_m\right) \cdot i_{qs} + x_m i'_{qr} - \gamma_q \cdot x_m i_{qs} + \gamma_q \cdot x_m i_{qs}$$

$$\lambda_{ds} = \left(x_{la} + k^2 x_m\right) \cdot i_{ds} + x_m i'_{dr} - \gamma_d \cdot k^2 x_m i_{ds} + \gamma_d \cdot k^2 x_m i_{ds}$$

(465-466)

$$\lambda'_{dR} = \gamma_d \cdot \lambda'_{dr} = \gamma_d \cdot k^2 x_m i_{ds} + \gamma_d \cdot k^2 \left(x_r + x_m\right) \cdot i'_{dr} - k^2 x_m i'_{dr} + k^2 x_m i'_{dr}$$

$$\lambda'_{qR} = \gamma_q \cdot \lambda'_{qr} = \gamma_q \cdot x_m i_{qs} + \gamma_q \cdot \left(x_r + x_m\right) \cdot i'_{qr} - x_m i'_{qr} + x_m i'_{qr}$$

(467-468)

The transformed rotor current $i'_{d(q)R}$ equals:

$$i'_{dR} = \frac{i'_{dr}}{\gamma_d}; \quad i'_{qR} = \frac{i'_{qr}}{\gamma_q}$$

(469)

The flux linkage equations reduce to:

$$\lambda_{ds} = \left(x_{la} + k^2 \left(1 - \gamma_d\right) \cdot x_m\right) \cdot i_{ds} + \gamma_d \cdot k^2 x_m \cdot \left(i_{ds} + i'_{dR}\right) =$$

$$= \left(x_{la} + k^2 \left(1 - \gamma_d\right) \cdot x_m\right) \cdot i_{ds} + \lambda'_{d\gamma_d}$$

$$\lambda_{qs} = \left(x_{lm} + \left(1 - \gamma_q\right) \cdot x_m\right) \cdot i_{qs} + \gamma_q \cdot x_m \cdot \left(i_{qs} + i'_{qR}\right) =$$

$$= \left(x_{lm} + \left(1 - \gamma_q\right) \cdot x_m\right) \cdot i_{qs} + \lambda'_{q\gamma_q}$$

$$\lambda'_{dR} = \gamma_d \cdot k^2 \left(\gamma_d \cdot x_r + \left(\gamma_d - 1\right) \cdot x_m\right) \cdot i'_{dR} + \gamma_d \cdot k^2 x_m \cdot \left(i_{ds} + i'_{dR}\right) =$$

$$= \lambda'_{d\gamma_d} + \gamma_d \cdot k^2 \left(\gamma_d \cdot x_r + \left(\gamma_d - 1\right) \cdot x_m\right) \cdot i'_{dR}$$

$$\lambda'_{qR} = \gamma_q \cdot \left(\gamma_q \cdot x_r + \left(\gamma_q - 1\right) \cdot x_m\right) \cdot i'_{qR} + \gamma_q \cdot x_m \cdot \left(i_{qs} + i'_{qR}\right) =$$

$$= \lambda'_{q\gamma_q} + \gamma_q \cdot \left(\gamma_q \cdot x_r + \left(\gamma_q - 1\right) \cdot x_m\right) \cdot i'_{qR}$$

(470-473)

This flux linkage can be represented by the equivalent circuit given in Fig. 3.10. The rotor quantities are now calculated with the equations:

$$0 = r'_R \, i'_{dR} + \frac{p}{\omega_b} \lambda'_{dR} + k \cdot \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

$$0 = k^2 r'_R \, i'_{qR} + \frac{p}{\omega_b} \lambda'_{qR} - \frac{1}{k} \cdot \frac{\omega_r}{\omega_b} \lambda'_{dR}$$

$$r'_R = \gamma_{q(d)}^2 \cdot r_r$$

(474-476)

Through variation of the arbitrary turn ratios "γ_q" and "γ_d" there are several options to select the appropriate reference vector flux for controlling the single-phase induction machine. Table 3.II summarises the available choices:
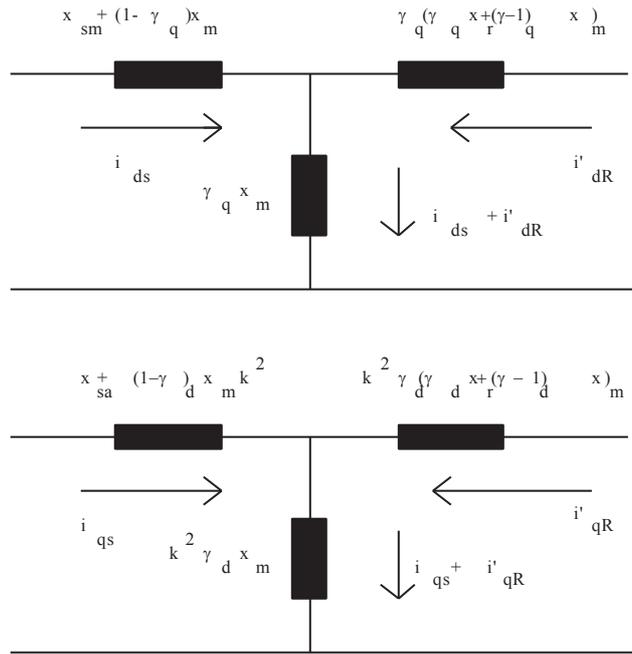
Fig. 3.10. Flux linkage equivalent circuit with arbitrary turn ratio
for unsymmetrical single-phase induction machine

TABLE 3.II
Specific choices of the turn ratio "γ_q" and "γ_d"

Turn ratio	Mathematical model	$\lambda_{d(q)\gamma}$	Flux selected
$\gamma_q = \dfrac{x_m}{x_m + x_r}$ $\gamma_d = \dfrac{k^2 x_m}{k^2(x_m + x_r)} = \gamma_q$	Inverse Γ-form	$\lambda_{d(q)r}$	**Rotor flux**
$\gamma_q = \gamma_d = 1$	T-form	$\lambda_{d(q)m}$	**Air-gap flux**
$\gamma_q = \dfrac{x_m + x_{lm}}{x_m}$ $\gamma_d = \dfrac{k^2 x_m + x_{la}}{k^2 x_m} \cong \gamma_q$	Γ-form	$\lambda_{d(q)s}$	**Stator flux**

The universal mathematical model given in Fig. 3.10 permits a choice between different flux vectors by selecting only two parameters "γ_q" and "γ_d" that have practically equal values, and denote the arbitrary turn ratio for each axis of the system. As in the symmetrical single-phase induction machine case, all other basic machine parameters remain unaffected.

3.7. D-Q axes models of the single-phase induction machine

The single-phase induction machine can be modelled by using different state-space variables, keeping as inputs the stator voltages and the load torque, and as outputs the electromagnetic torque and rotor angular velocity. Let us define the possible set of currents and flux linkages per second space vectors as follows:

$$\underline{i}_s = \underline{i}_{ds} + j \cdot i_{qs}$$
$$\underline{i}_r = i'_{dr} + j \cdot i'_{qr}$$
$$\underline{i}_m = \underline{i}_{dm} + j \cdot i_{qm}$$
$$\underline{\lambda}_s = \lambda_{ds} + j \cdot \lambda_{qs}$$
$$\underline{\lambda}'_r = \lambda'_{dr} + j \cdot \lambda'_{qr}$$
$$\underline{\lambda}_m = \lambda_{dm} + j \cdot \lambda_{qm}$$

$$(477\text{-}482)$$

$$\left[x\right] = \begin{bmatrix} \underline{i}_s & \underline{i}'_r & \underline{i}_m & \underline{\lambda}_s & \underline{\lambda}'_r & \underline{\lambda}_m \end{bmatrix}^T \tag{483}$$

Two-space vectors have to be selected from the six available vectors. Let the selected pair of state-space variables be denoted as $\underline{x}_1, \underline{x}_2$. It is possible to express the set of six state-space variables in terms of the two selected state-space variables as follows:

$$\left[x\right] = \begin{bmatrix} \underline{i}_s & \underline{i}'_r & \underline{i}_m & \underline{\lambda}_s & \underline{\lambda}'_r & \underline{\lambda}_m \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_1 & \underline{x}_2 \end{bmatrix}^T \tag{484}$$

where the coefficients a_{ij}, depend on the chosen set of state-space variables.

Out of fifteen possible state-space versions obtained from combining two space vectors from the total of six, only fourteen represent valid mathematical models for the single-phase induction machine. Please note that the pair of state-space variables that comprise the airgap flux linkage space vector and the magnetising current space vector cannot be selected as these vectors have the same direction. These models can be classified in three types: *currents space vectors, flux linkages space vectors* and *mixed currents-flux linkages space vectors models*.

In conjunction with the mechanical equation:

$$p\omega_r = \frac{P}{2J} \cdot \left(T_e - T_L\right) \tag{485}$$

we obtain a complete version of the single-phase induction machine model, viewed as the key for a motion control system.

For an unitary approach on all the possible models, and considering that the magnetising reactance has generally much greater value than the leakage reactance, the following approximation is made without altering the final results:

$$x_{la} + k^2 x_m \cong k^2 \left(x_{lm} + x_m\right) = k^2 x_s \tag{486}$$

The starting point for the state-variable models is given by voltage equations system written in stationary reference frame:

$$\boldsymbol{u} = \boldsymbol{L} \cdot p\boldsymbol{x} + \boldsymbol{R} \cdot \boldsymbol{x} \tag{487}$$

where: $\boldsymbol{x}$ is the selected set of state-variables and represent also the output of the model, $\boldsymbol{u}$ is the input vector (stator voltages), $\boldsymbol{L}$ is the coefficients matrix (it can be formed by reactance values, or non-dimensional elements) for multiplying the time derivative of the state-variables, $\boldsymbol{R}$ is the coefficients matrix (it can be formed by resistances and reactances values or non-dimensional elements) for multiplying the state-variables and p stand for the differential operator (d/dt). It results the general form of the state-variables system:

$$p\boldsymbol{x} = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{B} \cdot \boldsymbol{u} \tag{488}$$

where:

$$\boldsymbol{A} = -\boldsymbol{L}^{-1} \cdot \boldsymbol{R}$$
$$\boldsymbol{B} = \boldsymbol{L}^{-1} \tag{489-490}$$

All the following mathematical models can theoretically be discretised and used for implementation of controllers in the drive systems with single-phase induction machines. However, practically only few of them are suitable for this purpose as the computational effort, and implicitly the cost of the hardware, is a decisive factor to be considered.

When the vector control strategy has to be chosen, each of the six space vectors that can be selected as state-space variables represents a possible basis for a reference frame. The algorithm for implementing field oriented control systems is obtained as follows:

1. The rotor flux space vector is expressed as function of state-variables;
2. The rotor angular velocity ω_r term is substituted with ($\omega_r - \omega_k$) where ω_k is the angular velocity of the synchronous frame;
3. The dynamic equations that relates the rotor angular velocity to rotor flux linkages and voltages are deduced in the new state-variable system;
4. The synchronous reference frame is selected linked to one of the space vectors, which means that the q-axis component of the reference space vector is null;
5. The torque equation is computed according to the selected flux or current space vector in the synchronous reference frame.

3.7.1. Models with currents space vectors as state-space variables

1) The most used model of the single-phase induction machine is the currents space vectors state-space variables $\boldsymbol{x} = \left[i_{qs}, i_{ds}, i'_{qr}, i'_{dr} \right]^{\mathrm{T}}$. An important feature of this model is that the stator and leakage reactance, arbitrarily considered equal in many implementations, are now included in the stator and self-reactance values. As the stator currents are easily measurable quantities, a controller based on these state variables gives good performance accuracy. The model is readily available from the equations of voltages and flux linkages per second, and it can be expressed in a matrix form as follows:

$$\frac{p}{\omega_{\mathrm{b}}} \cdot \begin{bmatrix} i_{\mathrm{qs}} \\ i_{\mathrm{ds}} \\ i'_{\mathrm{qr}} \\ i'_{\mathrm{dr}} \end{bmatrix} = \begin{bmatrix} -\dfrac{x'_{\mathrm{r}} r_{\mathrm{m}}}{D} & -\dfrac{x_{\mathrm{m}}^2}{D} \cdot \dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & \dfrac{x_{\mathrm{m}} r'_{\mathrm{r}}}{D} & -\dfrac{x_{\mathrm{m}} x'_{\mathrm{r}}}{D} \cdot \dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} \\[3mm] \dfrac{x_{\mathrm{m}}^2}{D} \cdot \dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & -\dfrac{r_{\mathrm{a}} x'_{\mathrm{r}}}{D} \cdot \dfrac{1}{k^2} & \dfrac{x_{\mathrm{m}} x'_{\mathrm{r}}}{D} \cdot \dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & \dfrac{x_{\mathrm{m}} r'_{\mathrm{r}}}{D} \\[3mm] \dfrac{x_{\mathrm{m}} r_{\mathrm{m}}}{D} & \dfrac{x_{\mathrm{s}} x_{\mathrm{m}}}{D} \cdot \dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & -\dfrac{x_{\mathrm{s}} r'_{\mathrm{r}}}{D} & \dfrac{x_{\mathrm{s}} x'_{\mathrm{r}}}{D} \cdot \dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} \\[3mm] -\dfrac{x_{\mathrm{s}} x_{\mathrm{m}}}{D} \cdot \dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & \dfrac{x_{\mathrm{m}} r_{\mathrm{a}}}{D} \cdot \dfrac{1}{k^2} & -\dfrac{x_{\mathrm{s}} x'_{\mathrm{r}}}{D} \cdot \dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & -\dfrac{x_{\mathrm{s}} r'_{\mathrm{r}}}{D} \end{bmatrix} \cdot \begin{bmatrix} i_{\mathrm{qs}} \\ i_{\mathrm{ds}} \\ i'_{\mathrm{qr}} \\ i'_{\mathrm{dr}} \end{bmatrix} +$$

$$\begin{bmatrix} \dfrac{x'_{\mathrm{r}}}{D} & 0 & \dfrac{-x_{\mathrm{m}}}{D} & 0 \\[3mm] 0 & \dfrac{1}{k^2 x'_{\mathrm{r}} D} & 0 & -\dfrac{1}{k^2 x_{\mathrm{m}} D} \\[3mm] -\dfrac{x_{\mathrm{m}}}{D} & 0 & \dfrac{x_{\mathrm{s}}}{D} & 0 \\[3mm] 0 & -\dfrac{1}{k^2 x_{\mathrm{m}} D} & 0 & \dfrac{1}{k^2 x_{\mathrm{s}} D} \end{bmatrix} \cdot \begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix} \tag{491}$$

where:

$$D = x_{\mathrm{s}} x'_{\mathrm{r}} - x_{\mathrm{m}}^2$$

The electromagnetic torque can be computed as:

$$T_{\mathrm{e}} = \frac{P}{2} \cdot k \cdot \frac{x_{\mathrm{m}}}{\omega_{\mathrm{b}}} \cdot \left(i_{\mathrm{qs}} i'_{\mathrm{dr}} - i_{\mathrm{ds}} i'_{\mathrm{qr}} \right) \tag{492}$$

As it can be observed, a shortcoming for this model is the full system matrix, with all 16 elements non-zero values.

II) Another mathematical d-q axis model based on currents space vectors is the one which comprises the stator and the magnetising space vector currents as state-space variables $x = \left[i_{\mathrm{qs}}, i_{\mathrm{ds}}, i_{\mathrm{qm}}, i_{\mathrm{dm}} \right]^{\mathrm{T}}$. The state matrix elements are all non-zero, which implies a computational effort similar to the previously analysed model.

$$\frac{p}{\omega_b}\begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qm} \\ i_{dm}\end{bmatrix} = \begin{bmatrix} -\dfrac{r_m x'_r + r'_r x_m}{A_1} & \dfrac{x_m x'_{lr}}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} & \dfrac{x_m r'_r}{A_1} & -\dfrac{x_m x'_r}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} \\[2mm] -\dfrac{x_m x'_{lr}}{A_2}\cdot\dfrac{k\omega_r}{\omega_b} & -\dfrac{r_a x'_r + k^2 r'_r x_m}{A_2} & \dfrac{x_m x'_r}{A_2}\cdot\dfrac{k\omega_r}{\omega_b} & \dfrac{k^2 r'_r x_m}{A_2} \\[2mm] \dfrac{r'_r x_{lm} - r_m x'_{lr}}{A_1} & -\dfrac{x_{lm} x'_{lr}}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} & -\dfrac{r'_r x_{ls}}{A_1} & \dfrac{x_{lm} x'_r}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} \\[2mm] \dfrac{x_{la} x'_{lr}}{A_2}\cdot\dfrac{\omega_r}{k\omega_b} & \dfrac{r'_r x_{la} - r_a x'_{lr}}{A_2} & -\dfrac{x_{la} x'_r}{A_2}\cdot\dfrac{\omega_r}{k\omega_b} & -\dfrac{r'_r x_{la}}{A_2} \end{bmatrix}\cdot\begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qm} \\ i_{dm}\end{bmatrix} +$$

$$+ \begin{bmatrix} \dfrac{x'_r}{A_1} & 0 & -\dfrac{x_m}{A_1} & 0 \\[2mm] 0 & \dfrac{x'_r}{A_2} & 0 & -\dfrac{x_m}{A_2} \\[2mm] \dfrac{x'_{lr}}{A_1} & 0 & \dfrac{x_{ls}}{A_1} & 0 \\[2mm] 0 & \dfrac{x'_{lr}}{A_2} & 0 & \dfrac{x_{la}}{k^2 A_2}\end{bmatrix}\cdot\begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr}\end{bmatrix}$$

$$\tag{493}$$

where:

$$A_1 = x_{lm}x'_r + x_m x'_{lr} = D$$

$$A_2 = k^2 x_m x'_{lr} + x_{la} x'_r \cong k^2 D$$

The instantaneous electromagnetic torque may be expressed in terms of the state-space variables:

$$T_e = \frac{P}{2}\cdot k\cdot\frac{x_m}{\omega_b}\cdot\left(i_{qs} i_{dm} - i_{ds} i_{qm}\right) \tag{494}$$

III) The third possible combination of current space vectors as state-space variables is obtained by selecting magnetising and rotor currents $x = \left[i'_{qr}, i'_{dr}, i_{qm}, i_{dm}\right]^T$. When compared to the other current space vector models, the similar computing burden for obtaining the outputs of the system is obvious. The main difference between them is the presence of the global parameters (self-reactances) in the first model, while in the other two models an accurate determination of the leakage reactance is necessary. A complete description of the state-variables in matrix notation is given below:

$$\frac{p}{\omega_b}\begin{bmatrix} i'_{qr} \\ i'_{dr} \\ i_{qm} \\ i_{dm} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_m x_m + r'_r x_s}{A_1} & \dfrac{x_s x'_{lr}}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} & \dfrac{r_m x_m}{A_1} & \dfrac{x_s x_m}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} \\ -\dfrac{x_s x'_{lr}}{A_2}\cdot\dfrac{k\omega_r}{\omega_b} & -\dfrac{r_a x_m + k^2 r'_r x_s}{A_2} & -\dfrac{x_s x_m}{A_2}\cdot\dfrac{k\omega_r}{\omega_b} & \dfrac{r_a x_m}{A_2} \\ \dfrac{r_m x'_{lr} - r'_r x_{lm}}{A_1} & \dfrac{x_{lm} x'_{lr}}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} & -\dfrac{r_m x'_{lr}}{A_1} & \dfrac{x_{lm} x_m}{A_1}\cdot\dfrac{k\omega_r}{\omega_b} \\ -\dfrac{x_{la} x'_{lr}}{A_2}\cdot\dfrac{\omega_r}{k\omega_b} & \dfrac{r_a x'_{lr} - k^2 r'_r x_{la}}{A_2} & -\dfrac{x_{la} x_m}{A_2}\cdot\dfrac{\omega_r}{k\omega_b} & -\dfrac{r_a x'_{lr}}{A_2} \end{bmatrix}\cdot\begin{bmatrix} i'_{qr} \\ i'_{dr} \\ i_{qm} \\ i_{dm} \end{bmatrix} +$$

$$+ \begin{bmatrix} -\dfrac{x_m}{A_1} & 0 & \dfrac{x_s}{A_1} & 0 \\ 0 & -\dfrac{x_m}{A_2} & 0 & \dfrac{x_s}{A_2} \\ \dfrac{x'_{lr}}{A_1} & 0 & \dfrac{x_{lm}}{A_1} & 0 \\ 0 & \dfrac{x'_{lr}}{A_2} & 0 & \dfrac{x_{la}}{k^2 A_2} \end{bmatrix}\cdot\begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}$$

(495)

where:

$$A_1 = x_{lm} x_m + x'_{lr} x_s = D$$

$$A_2 = x_{la} x_m + k^2 x'_{lr} x_s \cong k^2 D$$

The instantaneous electromagnetic torque is expressed as follows:

$$T_e = \frac{P}{2}\cdot k\cdot\frac{x_m}{\omega_b}\cdot\left(i'_{dr} i_{qm} - i'_{qr} i_{dm} \right)$$

(496)

3.7.2. Models with flux linkages as state-space variables

I) In this case the flux linkages space vectors, $x = \left[\lambda_{qs}, \lambda_{ds}, \lambda'_{qr}, \lambda'_{dr} \right]^T$ are used for describing the mathematical model of the machine. The matrix equations for this system are as follows:

$$\frac{p}{\omega_b}\cdot\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} = \begin{bmatrix} \dfrac{-r_m x'_r}{D} & 0 & \dfrac{r_m x_m}{D} & 0 \\ 0 & \dfrac{-r_a x'_r}{k^2 D} & 0 & \dfrac{r_a x_m}{k^2 D} \\ \dfrac{r'_r x_m}{D} & 0 & \dfrac{-r'_r x_s}{D} & \dfrac{\omega_r}{k\omega_b} \\ 0 & \dfrac{r'_r x_m}{D} & -\dfrac{k\omega_r}{\omega_b} & \dfrac{-r'_r x_s}{D} \end{bmatrix}\cdot\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qr} \\ \lambda'_{dr} \end{bmatrix} + \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}$$

(497)

where D has the same significance as in the previous cases.

The electromagnetic torque is determined with the equivalent relation:

$$T_e = \frac{P}{2}\cdot\frac{x_m}{k\omega_b\cdot D}\cdot\left(\lambda_{qs}\lambda'_{dr} - \lambda_{ds}\lambda'_{qr} \right)$$

(498)

As opposed to the current models, it can be observed that the computational burden is substantially lower. Due to this important feature, this model is the most suitable for discretisation in motion control strategies.

II) An alternative to model the single-phase induction machine with flux space vectors as state-variable system, is the selection of air-gap flux space vector among the set of independent variables. A first approach is given by the stator and air-gap flux space vectors selected as state-space variables $x = \left[\lambda_{qs}, \lambda_{ds}, \lambda_{qm}, \lambda_{dm}\right]^T$.

$$
\frac{p}{\omega_b}\begin{bmatrix}\lambda_{qs}\\\lambda_{ds}\\\lambda_{qm}\\\lambda_{dm}\end{bmatrix} = \begin{bmatrix}
-\dfrac{r_m}{x_{lm}} & 0 & \dfrac{r_m}{x_{lm}} & 0 \\[2mm]
0 & -\dfrac{r_a}{x_{la}} & 0 & \dfrac{r_a}{x_{la}} \\[2mm]
-\dfrac{r_m x'_{lr} x_m}{x_{lm} A_1} + \dfrac{r'_r x_m}{A_1} & -\dfrac{x_{lm} x_m x'_{lr}}{x_{la} A_1} \cdot \dfrac{k\omega_r}{\omega_b} & \dfrac{r_m x_m x'_{lr}}{x_{lm} A_1} - \dfrac{r'_r x_s}{A_1} & \dfrac{x_{lm} x_m}{A_1} \cdot \dfrac{\omega_r}{k\omega_b} \cdot \left(1 + \dfrac{k^2 x'_{lr} x_s}{x_{la} x_m}\right) \\[2mm]
\dfrac{x_{la} x_m x'_{lr}}{x_{lm} A_2} \cdot \dfrac{k\omega_r}{\omega_b} & -\dfrac{r_a k^2 x'_{lr} x_m}{x_{la} A_2} + \dfrac{k^2 r'_r x_m}{A_2} & -\dfrac{x_{la} x_m}{A_2} \cdot \dfrac{k\omega_r}{\omega_b} \cdot \left(1 + \dfrac{x'_{lr} x_s}{x_{lm} x_m}\right) & \dfrac{r_a k^2 x'_{lr} x_m}{x_{la} A_2} - \dfrac{k^2 r'_r x_s}{A_2}
\end{bmatrix} \cdot \begin{bmatrix}\lambda_{qs}\\\lambda_{ds}\\\lambda_{qm}\\\lambda_{dm}\end{bmatrix} +
$$

$$
+ \begin{bmatrix}
1 & 0 & 0 & 0 \\[2mm]
0 & 1 & 0 & 0 \\[2mm]
\dfrac{x'_{lr} x_m}{A_1} & 0 & \dfrac{x_{lm} x_m}{A_1} & 0 \\[2mm]
0 & \dfrac{k^2 x'_{lr} x_m}{A_2} & 0 & \dfrac{x_{la} x_m}{A_2}
\end{bmatrix} \cdot \begin{bmatrix}u_{qs}\\u_{ds}\\u'_{qr}\\u'_{dr}\end{bmatrix}
$$

$$(499)$$

where:

$$A_1 = x_{lm} x_m + x'_{lr} x_s = D$$

$$A_2 = x_{la} x_m + k^2 x'_{lr} x_s \cong k^2 D$$

The electromagnetic torque expression shows that this model is prohibitive for implementation for modelling the single-phase induction machine in a vector control system and becomes:

$$T_e = \frac{P}{2} \cdot \frac{1}{\omega_b} \cdot \left[\lambda_{qs}\lambda_{ds} \cdot \left(\frac{1}{kx_{lm}} - \frac{k}{x_{la}}\right) + \frac{k}{x_{la}}\lambda_{qs}\lambda_{dm} - \frac{1}{kx_{lm}}\lambda_{dm}\lambda_{qm}\right] \tag{500}$$

which can be simplified only if the approximation $x_{la} = k^2 x_{lm}$ is reasonable to be made:

$$T_e = \frac{P}{2} \cdot \frac{1}{kx_{lm}\omega_b} \cdot \left[\lambda_{qs}\lambda_{dm} - \lambda_{dm}\lambda_{qm}\right] \tag{501}$$

III) The third option of selecting flux space vectors as state-space variables is the one which comprises the air-gap (magnetising) flux and rotor flux space vectors $x = \left[\lambda_{qm}, \lambda_{dm}, \lambda'_{qr}, \lambda'_{dr}\right]^T$. By comparison to the previously analysed model, this one presumes the same computational effort, but the electromagnetic torque is determined in a suitable form for vector control, without any supplementary approximations.

$$\frac{p}{\omega_{\mathrm{b}}}\begin{bmatrix} \lambda_{\mathrm{qm}} \\ \lambda_{\mathrm{dm}} \\ \lambda'_{\mathrm{qr}} \\ \lambda'_{\mathrm{dr}} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_{\mathrm{m}}x'_{\mathrm{r}}}{A_1}+\dfrac{r'_{\mathrm{r}}x_{\mathrm{m}}x_{\mathrm{lm}}}{x'_{\mathrm{lr}}A_1} & 0 & \dfrac{r_{\mathrm{m}}x_{\mathrm{m}}}{A_1}-\dfrac{r'_{\mathrm{r}}x_{\mathrm{m}}x_{\mathrm{lm}}}{x'_{\mathrm{lr}}A_1} & \dfrac{x_{\mathrm{m}}x_{\mathrm{lm}}}{A_1}\cdot\dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} \\[2mm] 0 & -\dfrac{r_{\mathrm{a}}x'_{\mathrm{r}}}{A_2}+\dfrac{r'_{\mathrm{r}}x_{\mathrm{m}}x_{\mathrm{la}}}{x'_{\mathrm{lr}}A_2} & -\dfrac{x_{\mathrm{m}}x_{\mathrm{la}}}{A_2}\cdot\dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & \dfrac{r_{\mathrm{a}}x_{\mathrm{m}}}{A_2}-\dfrac{r'_{\mathrm{r}}x_{\mathrm{m}}x_{\mathrm{la}}}{x'_{\mathrm{lr}}A_2} \\[2mm] \dfrac{r'_{\mathrm{r}}}{x'_{\mathrm{lr}}} & 0 & -\dfrac{r'_{\mathrm{r}}}{x'_{\mathrm{lr}}} & \dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} \\[2mm] 0 & \dfrac{r'_{\mathrm{r}}}{x'_{\mathrm{lr}}} & -\dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & -\dfrac{r'_{\mathrm{r}}}{x'_{\mathrm{lr}}} \end{bmatrix}\cdot\begin{bmatrix} \lambda_{\mathrm{qm}} \\ \lambda_{\mathrm{dm}} \\ \lambda'_{\mathrm{qr}} \\ \lambda'_{\mathrm{dr}} \end{bmatrix} +$$

$$+\begin{bmatrix} \dfrac{x_{\mathrm{m}}x'_{\mathrm{lr}}}{A_1} & 0 & \dfrac{x_{\mathrm{m}}x'_{\mathrm{lr}}}{A_1} & 0 \\[2mm] 0 & \dfrac{x_{\mathrm{m}}x'_{\mathrm{lr}}}{A_2} & 0 & \dfrac{x_{\mathrm{m}}x_{\mathrm{la}}}{A_2} \\[2mm] 0 & 0 & 1 & 0 \\[2mm] 0 & 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

$$\tag{502}$$

where:

$$A_1 = x_{\mathrm{m}}x'_{\mathrm{lr}}+x_{\mathrm{lm}}x'_{\mathrm{r}} = D$$

$$A_2 = k^2 x_{\mathrm{m}}x'_{\mathrm{lr}}+x_{\mathrm{lm}}x'_{\mathrm{r}} \cong k^2 D$$

The resulting relation for computing instantaneous electromagnetic torque is as follows:

$$T_{\mathrm{e}} = \frac{P}{2}\cdot\frac{1}{kx'_{\mathrm{lr}}\omega_{\mathrm{b}}}\cdot\left(\lambda_{\mathrm{qm}}\lambda'_{\mathrm{dr}}-\lambda_{\mathrm{dm}}\lambda'_{\mathrm{qr}}\right) \tag{503}$$

3.7.3. Models with mixed currents – flux space vectors state-space variables

I) If the stator variables are chosen for modelling the single-phase induction machine system, then a mixed flux linkages-currents state-space variables model $x = \left[\lambda_{\mathrm{qs}},\lambda_{\mathrm{ds}},i_{\mathrm{qs}},i_{\mathrm{ds}}\right]^{\mathrm{T}}$ is developed. The matrix equations and the electromagnetic torque relation are:

$$\frac{p}{\omega_{\mathrm{b}}}\begin{bmatrix} \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \\ i_{\mathrm{qs}} \\ i_{\mathrm{ds}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -r_{\mathrm{m}} & 0 \\[2mm] 0 & 0 & 0 & -r_{\mathrm{a}} \\[2mm] \dfrac{r'_{\mathrm{r}}}{D} & -\dfrac{x'_{\mathrm{r}}}{D}\cdot\dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & -\dfrac{r_{\mathrm{m}}x'_{\mathrm{r}}+r'_{\mathrm{r}}x_{\mathrm{s}}}{D} & \dfrac{k\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} \\[2mm] \dfrac{x'_{\mathrm{r}}}{D}\cdot\dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & \dfrac{r'_{\mathrm{r}}}{k^2 D} & -\dfrac{\omega_{\mathrm{r}}}{k\omega_{\mathrm{b}}} & -\dfrac{r_{\mathrm{m}}x'_{\mathrm{r}}+k^2 r'_{\mathrm{r}}x_{\mathrm{s}}}{k^2 D} \end{bmatrix}\cdot\begin{bmatrix} \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \\ i_{\mathrm{qs}} \\ i_{\mathrm{ds}} \end{bmatrix}+\begin{bmatrix} 1 & 0 & 0 & 0 \\[2mm] 0 & 1 & 0 & 0 \\[2mm] \dfrac{x'_{\mathrm{r}}}{D} & 0 & 0 & 0 \\[2mm] 0 & \dfrac{x'_{\mathrm{r}}}{k^2 D} & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

$$\tag{504}$$

$$T_{\mathrm{e}} = \frac{P}{2}\cdot\frac{1}{\omega_{\mathrm{b}}}\left(\frac{1}{k}\lambda_{\mathrm{ds}}i_{\mathrm{qs}}-k\lambda_{\mathrm{qs}}i_{\mathrm{ds}}\right) \tag{505}$$

II) Another important mixed flux linkages-currents state-space variables model $x = \left[\lambda'_{\mathrm{qr}},\lambda'_{\mathrm{dr}},i'_{\mathrm{qr}},i'_{\mathrm{dr}}\right]^{\mathrm{T}}$ is that expressed in rotor quantities. This model can be easily used for the rotor flux oriented control of the machine. We can describe the system as follows:

$$\frac{p}{\omega_b}\begin{bmatrix}\lambda'_{qr}\\\lambda'_{dr}\\i'_{qr}\\i'_{dr}\end{bmatrix} = \begin{bmatrix}0 & \dfrac{\omega_r}{k\omega_b} & -r'_r & 0\\[2mm] -\dfrac{k\omega_r}{\omega_b} & 0 & 0 & -k^2 r'_r\\[2mm] \dfrac{r_m}{D} & \dfrac{x_s}{D}\cdot\dfrac{\omega_r}{\omega_b} & -\dfrac{r_m x'_r + r'_r x_s}{D} & 0\\[2mm] -\dfrac{x_s}{D}\cdot\dfrac{\omega_r}{k\omega_b} & \dfrac{r_a}{k^4 D} & 0 & -\dfrac{r_a x'_r + k^2 r'_r x_s}{k^2 D}\end{bmatrix}\cdot\begin{bmatrix}\lambda'_{qr}\\\lambda'_{dr}\\i'_{qr}\\i'_{dr}\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\-\dfrac{x_m}{D}&0&0&0\\0&-\dfrac{x_m}{k^2 D}&0&0\end{bmatrix}\cdot\begin{bmatrix}u_{qs}\\u_{ds}\\u'_{qr}\\u'_{dr}\end{bmatrix}$$

(506)

The electromagnetic torque relation is given below:

$$T_e = \frac{P}{2}\cdot\frac{1}{\omega_b}\left(k\lambda'_{qr}\, i'_{dr} - \frac{1}{k}\lambda'_{dr}\, i'_{qr}\right) \tag{507}$$

III) A similar model to the previous one is that with stator current and rotor flux linkage $x = \left[\lambda'_{qr},\lambda'_{dr},i_{qs},i_{ds}\right]^{\mathrm{T}}$ as state-space variables. It represents an alternative model for rotor flux oriented control strategies. The matrix equation and electromagnetic torque relation are presented below:

$$\frac{p}{\omega_b}\begin{bmatrix}i_{qs}\\i_{ds}\\\lambda'_{qr}\\\lambda'_{dr}\end{bmatrix} = \begin{bmatrix} -\dfrac{r_m x'^2_r + r'_r x^2_m}{x'_r D} & 0 & \dfrac{r'_r x_m}{x'_r D} & -\dfrac{x_m}{D}\cdot\dfrac{\omega_r}{k\omega_b}\\[3mm] 0 & -\dfrac{r_a x'^2_r + k^2 r'_r x^2_m}{k^2 x'_r D} & \dfrac{x_m}{D}\cdot\dfrac{k\omega_r}{\omega_b} & \dfrac{r'_r x_m}{k^2 x'_r D}\\[3mm] \dfrac{r'_r x_m}{x'_r} & 0 & -\dfrac{r'_r}{x'_r} & \dfrac{\omega_r}{k\omega_b}\\[3mm] 0 & \dfrac{k^2 r'_r x_m}{x'_r} & -\dfrac{k\omega_r}{\omega_b} & -\dfrac{r'_r}{x'_r}\end{bmatrix}\cdot\begin{bmatrix}i_{qs}\\i_{ds}\\\lambda'_{qr}\\\lambda'_{dr}\end{bmatrix} +$$

$$+\begin{bmatrix}\dfrac{x'_r}{D} & 0 & -\dfrac{x_m}{D} & 0\\[3mm] 0 & \dfrac{x'_r}{k^2 D} & 0 & -\dfrac{x_m}{k^2 D}\\[3mm] 0 & 0 & 1 & 0\\[3mm] 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}u_{qs}\\u_{ds}\\u'_{qr}\\u'_{dr}\end{bmatrix}$$

(508)

and torque expression is:

$$T_e = \frac{P}{2}\cdot\frac{x_m}{x'_r\,\omega_b}\cdot\left(\frac{1}{k}\cdot i_{qs}\lambda'_{dr} - k\cdot i_{ds}\lambda'_{qr}\right) \tag{509}$$

IV) A rarely used mathematical model is that with mixed rotor currents space vector and stator flux linkages space vector as state-space variables $x = \left[\lambda_{qs},\lambda_{ds},i'_{qr},i'_{dr}\right]^{\mathrm{T}}$. However, the advantage of this model is that merely only stator winding parameters are necessary, so the influence of rotor parameters is minimised. It can be used for an unconventional stator flux oriented control with rotor current components producing the torque and the flux. The expressions for implementing this model are:

$$\frac{p}{\omega_{\mathrm b}}\begin{bmatrix} i'_{\mathrm{qr}} \\ i'_{\mathrm{dr}} \\ \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_{\mathrm m}x_{\mathrm m}^2 + r'_{\mathrm r}x_{\mathrm s}^2}{x_{\mathrm s}D} & \dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & \dfrac{r_{\mathrm m}x_{\mathrm m}}{x_{\mathrm s}D} & \dfrac{x_{\mathrm m}}{D}\cdot\dfrac{\omega_{\mathrm r}}{k\omega_{\mathrm b}} \\[2mm] -\dfrac{\omega_{\mathrm r}}{k\omega_{\mathrm b}} & -\dfrac{r_{\mathrm a}x_{\mathrm m}^2 + k^2 r'_{\mathrm r}x_{\mathrm s}^2}{k^2 x_{\mathrm s}D} & -\dfrac{x_{\mathrm m}}{D} & \dfrac{r_{\mathrm a}x_{\mathrm m}}{k^4 D x_{\mathrm s}} \\[2mm] \dfrac{r_{\mathrm m}x_{\mathrm m}}{x_{\mathrm s}} & 0 & -\dfrac{r_{\mathrm m}}{x_{\mathrm s}} & 0 \\[2mm] 0 & \dfrac{r_{\mathrm a}x_{\mathrm m}}{x_{\mathrm s}} & 0 & -\dfrac{r_{\mathrm a}}{k^2 x_{\mathrm s}} \end{bmatrix}\cdot\begin{bmatrix} i'_{\mathrm{qr}} \\ i'_{\mathrm{dr}} \\ \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \end{bmatrix} + \begin{bmatrix} \dfrac{x_{\mathrm m}}{D} & 0 & \dfrac{x_{\mathrm s}}{D} & 0 \\[2mm] 0 & \dfrac{x_{\mathrm m}}{k^2 D} & 0 & \dfrac{x_{\mathrm s}}{k^2 D} \\[2mm] 1 & 0 & 0 & 0 \\[2mm] 0 & 1 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

(510)

where the significance of D was already stated.

The main system output, the electromagnetic torque, can be computed as:

$$T_{\mathrm e} = \frac{P}{2}\cdot\frac{x_{\mathrm m}}{x_{\mathrm s}\omega_{\mathrm b}}\cdot\left(k\cdot\lambda_{\mathrm{qs}}i'_{\mathrm{dr}} - \frac{1}{k}\cdot\lambda_{\mathrm{ds}}i'_{\mathrm{qr}}\right)$$

(511)

V) The mixed stator current space vector and air-gap flux space vector as state-space variables $x = \left[\lambda_{\mathrm{qm}}, \lambda_{\mathrm{dm}}, i_{\mathrm{qs}}, i_{\mathrm{ds}}\right]^{\mathrm T}$ belongs to one of the more complex model types. It preserves information regarding both stator and rotor parameters. Different from the previous mixed models, the state-space matrix contains only non-zero elements, which leads to greater computational effort.

$$\frac{p}{\omega_{\mathrm b}}\begin{bmatrix} i_{\mathrm{qs}} \\ i_{\mathrm{ds}} \\ \lambda_{\mathrm{qm}} \\ \lambda_{\mathrm{dm}} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_{\mathrm m}x'_{\mathrm r} + r'_{\mathrm r}x_{\mathrm m}}{A_1} & \dfrac{x_{\mathrm m}x'_{\mathrm{lr}}}{A_1}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & \dfrac{r'_{\mathrm r}}{A_1} & -\dfrac{x'_{\mathrm r}}{A_1}\cdot\dfrac{\omega_{\mathrm r}}{k\omega_{\mathrm b}} \\[2mm] \dfrac{x_{\mathrm m}x'_{\mathrm{lr}}}{A_2}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & -\dfrac{r_{\mathrm a}x'_{\mathrm r} + k^2 r'_{\mathrm r}x_{\mathrm m}}{A_2} & \dfrac{x'_{\mathrm r}}{A_2}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & \dfrac{r'_{\mathrm r}}{A_2} \\[2mm] \dfrac{x_{\mathrm m}(r'_{\mathrm r}x_{\mathrm m} - r_{\mathrm m}x'_{\mathrm{lr}})}{A_1} & -\dfrac{x_{\mathrm{lm}}x_{\mathrm m}x'_{\mathrm{lr}}}{A_1}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & -\dfrac{x_{\mathrm{lm}}r'_{\mathrm r}}{A_1} & \dfrac{x_{\mathrm{lm}}x'_{\mathrm r}}{A_1}\cdot\dfrac{\omega_{\mathrm r}}{k\omega_{\mathrm b}} \\[2mm] -\dfrac{x_{\mathrm{la}}x_{\mathrm m}x'_{\mathrm{lr}}}{A_2}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & \dfrac{x_{\mathrm m}k^2(-x'_{\mathrm{lr}}r_{\mathrm a} + x_{\mathrm{la}}r'_{\mathrm r})}{A_2} & -\dfrac{x_{\mathrm{la}}x'_{\mathrm r}}{A_2}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & -\dfrac{x_{\mathrm{la}}r'_{\mathrm r}}{A_2} \end{bmatrix}\cdot\begin{bmatrix} i_{\mathrm{qs}} \\ i_{\mathrm{ds}} \\ \lambda_{\mathrm{qm}} \\ \lambda_{\mathrm{dm}} \end{bmatrix} +$$

$$+\begin{bmatrix} \dfrac{x'_{\mathrm r}}{A_1} & 0 & -\dfrac{x_{\mathrm m}}{A_1} & 0 \\[2mm] 0 & \dfrac{x'_{\mathrm r}}{A_2} & 0 & \dfrac{x_{\mathrm m}}{A_2} \\[2mm] \dfrac{x'_{\mathrm{lr}}x_{\mathrm m}}{A_1} & 0 & \dfrac{x_{\mathrm{lm}}x_{\mathrm m}}{A_1} & 0 \\[2mm] 0 & \dfrac{k^2 x'_{\mathrm{lr}}x_{\mathrm m}}{A_2} & 0 & \dfrac{x_{\mathrm{la}}x_{\mathrm m}}{A_2} \end{bmatrix}\cdot\begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

(512)

where:

$$A_1 = x_{\mathrm{lm}}x'_{\mathrm r} + x'_{\mathrm{lr}}x_{\mathrm m} = D$$

$$A_2 = k^2 x'_{\mathrm{lr}}x_{\mathrm m} + x_{\mathrm{la}}x'_{\mathrm r} \cong k^2 D$$

The electromagnetic torque is computed as:

$$T_e = \frac{P}{2} \cdot \frac{1}{\omega_b} \cdot \left(\frac{1}{k} \cdot i_{qs} \lambda_{dm} - k \cdot i_{ds} \lambda_{qm} \right) \tag{513}$$

VI) A similar model to the precedent one, is the mixed rotor current space vector and air-gap flux space vector as state-space variables $x = \left[\lambda_{qm}, \lambda_{dm}, i'_{qr}, i'_{dr} \right]^T$. The matrix equation of the system is given below:

$$
\frac{p}{\omega_b} \begin{bmatrix} i'_{qr} \\ i'_{dr} \\ \lambda_{qm} \\ \lambda_{dm} \end{bmatrix} = \begin{bmatrix} -\dfrac{r_m x_m + r'_r x_s}{A_1} & \dfrac{x_s x'_{lr}}{A_1} \cdot \dfrac{k\omega_r}{\omega_b} & \dfrac{r_m}{A_1} & \dfrac{x_m}{A_1} \cdot \dfrac{\omega_r}{k\omega_b} \\ -\dfrac{x_s x'_{lr}}{A_2} \cdot \dfrac{k\omega_r}{\omega_b} & -\dfrac{r_a x_m + k^2 r'_r x_s}{A_2} & -\dfrac{x_s}{A_2} \cdot \dfrac{k\omega_r}{\omega_b} & -\dfrac{r_a}{k^2 A_2} \\ \dfrac{x_m(-r'_r x_{lm} + r_m x'_{lr})}{A_1} & \dfrac{x_{lm} x_m x'_{lr}}{A_1} \cdot \dfrac{k\omega_r}{\omega_b} & -\dfrac{x_{lm} r_m}{A_1} & \dfrac{x_{lm} x_m}{A_1} \cdot \dfrac{\omega_r}{k\omega_b} \\ -\dfrac{x_{la} x_m x'_{lr}}{A_2} \cdot \dfrac{k\omega_r}{\omega_b} & \dfrac{x_m(k^2 x'_{lr} r_a - k^2 x_{la} r'_r)}{A_2} & -\dfrac{x_{la} x_m}{A_2} \cdot \dfrac{k\omega_r}{\omega_b} & \dfrac{x_{la} r_a}{A_2} \end{bmatrix} \cdot \begin{bmatrix} i_{qs} \\ i_{ds} \\ \lambda_{qm} \\ \lambda_{dm} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} -\dfrac{x_m}{A_1} & 0 & \dfrac{x_s}{A_1} & 0 \\ 0 & -\dfrac{x_m}{A_2} & 0 & \dfrac{x_s}{A_2} \\ \dfrac{x'_{lr} x_m}{A_1} & 0 & \dfrac{x_{lm} x_m}{A_1} & 0 \\ 0 & \dfrac{k^2 x'_{lr} x_m}{A_2} & 0 & \dfrac{x_{la} x_m}{A_2} \end{bmatrix} \cdot \begin{bmatrix} u_{qs} \\ u_{ds} \\ u'_{qr} \\ u'_{dr} \end{bmatrix}
$$

$$\tag{514}$$

where:

$$A_1 = x_{lm} x_m + x'_{lr} x_s = D$$

$$A_2 = x_{la} x_m + k^2 x_{lr} x_s \cong k^2 D$$

The electromagnetic torque relation becomes:

$$T_e = \frac{P}{2} \cdot \frac{1}{\omega_b} \cdot \left(k \cdot i'_{dr} \lambda_{qm} - \frac{1}{k} \cdot i'_{qr} \lambda_{dm} \right) \tag{515}$$

VII) If the magnetising current space vector is selected as state-space variable together with one of flux linkages space vectors $x = \left[i_{qm}, i_{dm}, \lambda_{qs}, \lambda_{ds} \right]^T$, the state matrix computation gives several null elements. Nevertheless, the torque expression is more complicated and the final relation can be written in the same form as in previous cases only through some new assumptions,.

$$\frac{p}{\omega_{\mathrm b}}\begin{bmatrix} i_{\mathrm{qm}} \\ i_{\mathrm{dm}} \\ \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \end{bmatrix} = \begin{bmatrix} \dfrac{r_{\mathrm m}x'_{\mathrm{lr}}x_{\mathrm m} - r'_{\mathrm r}x_{\mathrm s}x_{\mathrm{ls}}}{x_{\mathrm{ls}}A_{\mathrm l}} & \dfrac{x_{\mathrm{lm}}A_2}{x_{\mathrm{la}}A_{\mathrm l}}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} & \dfrac{r'_{\mathrm r}x_{\mathrm{lm}} - r_{\mathrm m}x'_{\mathrm{lr}}}{x_{\mathrm{ls}}A_{\mathrm l}} & -\dfrac{x_{\mathrm{lm}}x'_{\mathrm{lr}}}{x_{\mathrm{la}}A_{\mathrm l}}\cdot\dfrac{k\omega_{\mathrm r}}{\omega_{\mathrm b}} \\[2.5ex] -\dfrac{x_{\mathrm{la}}A_{\mathrm l}}{x_{\mathrm{lm}}A_2}\cdot\dfrac{\omega_{\mathrm r}}{k\omega_{\mathrm b}} & \dfrac{k^2(r_{\mathrm a}x_{\mathrm m}x'_{\mathrm{lr}} - r'_{\mathrm r}x_{\mathrm s}x_{\mathrm{la}})}{x_{\mathrm{la}}A_2} & \dfrac{x_{\mathrm{la}}x'_{\mathrm{lr}}}{x_{\mathrm{lm}}A_2}\cdot\dfrac{\omega_r}{k\omega_b} & \dfrac{r'_{\mathrm r}x_{\mathrm{la}} - r_{\mathrm a}x'_{\mathrm{lr}}}{x_{\mathrm{la}}A_2} \\[2.5ex] \dfrac{r_{\mathrm m}x_{\mathrm m}}{x_{\mathrm{lm}}} & 0 & -\dfrac{r_{\mathrm m}}{x_{\mathrm{lm}}} & 0 \\[2.5ex] 0 & \dfrac{k^2 r_{\mathrm a}x_{\mathrm m}}{x_{\mathrm{la}}} & 0 & -\dfrac{r_{\mathrm a}}{x_{\mathrm{la}}} \end{bmatrix} \cdot \begin{bmatrix} i_{\mathrm{qm}} \\ i_{\mathrm{dm}} \\ \lambda_{\mathrm{qs}} \\ \lambda_{\mathrm{ds}} \end{bmatrix} +$$

$$+ \begin{bmatrix} \dfrac{x'_{\mathrm{lr}}}{A_{\mathrm l}} & 0 & \dfrac{x_{\mathrm{lm}}}{A_{\mathrm l}} & 0 \\[2ex] 0 & \dfrac{x'_{\mathrm{lr}}}{A_2} & 0 & \dfrac{x_{\mathrm{la}}}{k^2 A_2} \\[2ex] 1 & 0 & 0 & 0 \\[2ex] 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{\mathrm{qs}} \\ u_{\mathrm{ds}} \\ u'_{\mathrm{qr}} \\ u'_{\mathrm{dr}} \end{bmatrix}$$

(516)

where:

$$A_{\mathrm l} = x_{\mathrm m}x_{\mathrm{lm}} + x_{\mathrm s}x'_{\mathrm{lr}} = D$$

$$A_2 = x_{\mathrm m}x_{\mathrm{la}} + k^2 x_{\mathrm s}x'_{\mathrm{lr}} \cong k^2 D$$

The electromagnetic torque relation, with the approximation: $x_{\mathrm{la}} \cong k^2 x_{\mathrm{lm}}$ can be written as:

$$T_{\mathrm e} = \frac{P}{2}\cdot\frac{1}{\omega_{\mathrm b}}\cdot\left[x_{\mathrm m}\cdot\left(\frac{k^3}{x_{\mathrm{la}}}\cdot i_{\mathrm{dm}}\lambda_{\mathrm{qs}} - \frac{1}{kx_{\mathrm{lm}}}\cdot i_{\mathrm{qm}}\lambda_{\mathrm{ds}}\right) + \lambda_{\mathrm{qs}}\lambda_{\mathrm{ds}}\left(\frac{1}{kx_{\mathrm{lm}}} - \frac{k}{x_{\mathrm{la}}}\right)\right] \cong$$

$$\cong \frac{P}{2}\cdot\frac{x_{\mathrm m}}{\omega_{\mathrm b}x_{\mathrm{lm}}}\cdot\left(k\cdot i_{\mathrm{dm}}\lambda_{\mathrm{qs}} - \frac{1}{k}\cdot i_{\mathrm{qm}}\lambda_{\mathrm{ds}}\right)$$

(517)

The presence of oscillating terms given by the product between flux linkages, makes this model prohibitive for implementing in vector control strategies.

VIII) The last mixed current-flux linkages space vector model realises the connection between the magnetising current space-vector and the rotor flux linkage space vector $x = \left[i_{\mathrm{qm}}, i_{\mathrm{dm}}, \lambda'_{\mathrm{qr}}, \lambda'_{\mathrm{dr}}\right]^{\mathrm T}$, selected as state-space variables. The state matrix contains the same number of zero elements (four) and the instantaneous electromagnetic torque is determined in a suitable form for vector control implementation.

$$
\frac{p}{\omega_{\mathrm{b}}}
\begin{bmatrix}
i_{\mathrm{qm}} \\
i_{\mathrm{dm}} \\
\lambda'_{\mathrm{qr}} \\
\lambda'_{\mathrm{dr}}
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{r'_{\mathrm{r}} x_{\mathrm{lm}} x_{\mathrm{m}} - r_{\mathrm{m}} x'_{\mathrm{r}} x'_{\mathrm{lr}}}{x'_{\mathrm{lr}} A_1} & 0 & \dfrac{r_{\mathrm{m}} x'_{\mathrm{lr}} - r'_{\mathrm{r}} x_{\mathrm{lm}}}{x'_{\mathrm{lr}} A_1} & \dfrac{x_{\mathrm{lm}}}{A_1} \cdot \dfrac{\omega_{\mathrm{r}}}{k \omega_{\mathrm{b}}} \\[3mm]
0 & \dfrac{r'_{\mathrm{r}} x_{\mathrm{la}} x_{\mathrm{m}} - r_{\mathrm{a}} x'_{\mathrm{r}} x'_{\mathrm{lr}}}{x'_{\mathrm{lr}} A_2} & -\dfrac{x_{\mathrm{la}}}{A_2} \cdot \dfrac{\omega_{\mathrm{r}}}{k \omega_{\mathrm{b}}} & \dfrac{r_{\mathrm{a}} x'_{\mathrm{lr}} - x_{\mathrm{la}} r'_{\mathrm{r}}}{k^2 x'_{\mathrm{lr}} A_2} \\[3mm]
\dfrac{r'_{\mathrm{r}} x_{\mathrm{m}}}{x'_{\mathrm{lr}}} & 0 & -\dfrac{r'_{\mathrm{r}}}{x'_{\mathrm{lr}}} & \dfrac{\omega_{\mathrm{r}}}{k \omega_{\mathrm{b}}} \\[3mm]
0 & \dfrac{k^2 r'_{\mathrm{r}} x_{\mathrm{m}}}{x'_{\mathrm{lr}}} & -k \dfrac{\omega_{\mathrm{r}}}{\omega_{\mathrm{b}}} & -\dfrac{r'_{\mathrm{r}}}{x'_{\mathrm{lr}}}
\end{bmatrix}
\cdot
\begin{bmatrix}
i_{\mathrm{qm}} \\
i_{\mathrm{dm}} \\
\lambda'_{\mathrm{qr}} \\
\lambda'_{\mathrm{dr}}
\end{bmatrix}
+
$$

$$
+
\begin{bmatrix}
\dfrac{x'_{\mathrm{lr}}}{A_1} & 0 & \dfrac{x_{\mathrm{lm}}}{A_1} & 0 \\[3mm]
0 & \dfrac{x'_{\mathrm{lr}}}{A_2} & 0 & \dfrac{x_{\mathrm{la}}}{k^2 A_2} \\[3mm]
0 & 0 & 1 & 0 \\[3mm]
0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
u_{\mathrm{qs}} \\
u_{\mathrm{ds}} \\
u'_{\mathrm{qr}} \\
u'_{\mathrm{dr}}
\end{bmatrix}
$$

$$(518)$$

where:

$$A_1 = x_m x'_{lr} + x_{ls} x'_r = D$$

$$A_2 = k^2 x_m x'_{lr} + x_{la} x'_r \cong k^2 D$$

The electromagnetic torque is determined as follows:

$$T_{\mathrm{e}} = \frac{P}{2} \cdot \frac{x_{\mathrm{m}}}{\omega_{\mathrm{b}} x'_{\mathrm{lr}}} \cdot \left(\frac{1}{k} \cdot i_{\mathrm{qm}} \lambda'_{\mathrm{dr}} - k \cdot i_{\mathrm{dm}} \lambda'_{\mathrm{qr}} \right) \tag{519}$$

3.8. Vector control strategies for single-phase induction machine

The aim of vector control is usually to decouple the stator current i_s into its flux producing and torque producing components (i_{ds}, i_{qs} respectively) in order to obtain a decoupled control of the flux and the electromagnetic torque. For this reason a special reference frame is selected fixed to different space vector variables. The reference frame has to be synchronous, as all the space vectors have the same angular velocity given by the supply voltage frequency.

As the stator windings of the single-phase induction motor are usually unsymmetrical, the vector control principles have to be implemented in a special way. The machine parameters differ from axis d to axis q. The waveform of the electromagnetic torque demonstrates the unbalance of the system. Even for equal amplitude, orthogonal stator currents i_{qS} and i_{dS}, the torque contains AC terms. A new torque relation has to be developed for each case of vector control strategy of single-phase induction machine (Correa et al. - 1998, 1999).

Generally, the considerations made for the three-phase induction machine when the vector control sheme has to be developed, are valid. An easy to follow steps algorithm for implementing vector oriented control systems is obtained as follows:

1. A complete mathematical model of the single-phase induction machine is developed in stationary reference frame, the only reference frame that maintains constant parameters, according to the chosen set of state-space variables;
2. A suitable torque expression is determined, in order to eliminate the influence of AC terms;
3. The rotor based variables are completely expressed in the new state-variable system;
4. The rotor angular velocity ω_r term is substituted with $\omega_s = (\omega_r - \omega_k)$ where ω_k and ω_s are the angular velocity of the synchronous frame, respectively the slip angular velocity;

5. The synchronous reference frame is selected linked to one of the space vectors, which means that the q-axis component of the reference space vector is null;
6. The torque equation is computed according to the selected flux or current space vector in the synchronous reference frame.

The transformation of reference frames for the induction motor vector control can be summarised as shown in Fig. 3.11.
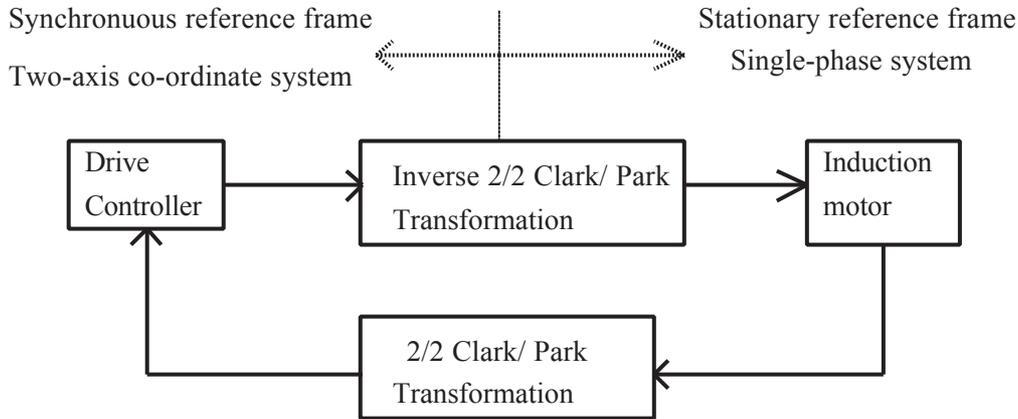
Synchronuous reference frame

Stationary reference frame
Single-phase system

Two-axis co-ordinate system

```
┌──────────────┐      ┌──────────────────────┐      ┌──────────────┐
│ Drive        │ ───> │ Inverse 2/2 Clark/ Park│ ───> │ Induction    │
│ Controller   │      │ Transformation         │      │ motor        │
└──────────────┘      └──────────────────────┘      └──────────────┘
        ^                                                    │
        │              ┌──────────────────────┐             │
        └───────────── │ 2/2 Clark/ Park        │ <──────────┘
                       │ Transformation         │
                       └──────────────────────┘
```

Fig. 3.11. Block diagram of transformation of frames and co-ordinate systems
for single-phase induction motor vector control

3.8.1. Stator field orientation (SFO)

For this vector control strategy, the set of state-space variables formed by stator flux linkage and current space vectors $x = \left[\lambda_{qs}, \lambda_{ds}, i_{qs}, i_{ds} \right]^{\mathrm{T}}$ is selected. The mathematical model is given in the chapter dedicated to *d-q* models of the single-phase induction machine.

The rotor based variables expressed in state-space variables terms are:

$$i\,^{\mathrm{s}}_{qr} = \frac{1}{x_{\mathrm{m}}}\left(\lambda^{\mathrm{s}}_{qs} - x_{\mathrm{s}} i^{\mathrm{s}}_{qs} \right)$$

$$i\,^{\mathrm{s}}_{dr} = \frac{\lambda^{\mathrm{s}}_{ds}}{k^2 x_{\mathrm{m}}} - \frac{x_{\mathrm{s}} i^{\mathrm{s}}_{ds}}{x_{\mathrm{m}}}$$

$$\lambda\,^{\mathrm{s}}_{qr} = \frac{1}{x_{\mathrm{m}}}\left(x'_{\mathrm{r}} \lambda^{\mathrm{s}}_{qs} - D \cdot i^{\mathrm{s}}_{qs} \right)$$

$$\lambda\,^{\mathrm{s}}_{dr} = \frac{1}{x_{\mathrm{m}}}\left(x'_{\mathrm{r}} \lambda^{\mathrm{s}}_{ds} - k^2 D \cdot i^{\mathrm{s}}_{ds} \right)$$

(520-523)

To obtain the machine equations in the synchronuous stator flux reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the stator flux to be zero. The stator voltages equations remain unchanged. If a cage rotor is considered, the resultant equations are as follows:

$$\left(r'_{\mathrm{r}}\, x_{\mathrm{s}} + D \frac{p}{\omega_{\mathrm{b}}} \right) i^{\mathrm{s}}_{qs} - \frac{\omega_{\mathrm{s}}}{\omega_{\mathrm{b}}} \left(\frac{1}{k} x'_{\mathrm{r}} \lambda^{\mathrm{s}}_{ds} - k \cdot D \cdot i^{\mathrm{s}}_{ds} \right) - \left(r'_{\mathrm{r}} + x'_{\mathrm{r}} \frac{p}{\omega_{\mathrm{b}}} \right) \lambda^{\mathrm{s}}_{qs} = 0$$

$$k^2 \left(r'_{\mathrm{r}}\, x_{\mathrm{s}} + D \frac{p}{\omega_{\mathrm{b}}} \right) i^{\mathrm{s}}_{ds} - k \frac{\omega_{\mathrm{s}}}{\omega_{\mathrm{b}}} D \cdot i^{\mathrm{s}}_{qs} = \left(r'_{\mathrm{r}} + x'_{\mathrm{r}} \frac{p}{\omega_{\mathrm{b}}} \right) \lambda^{\mathrm{s}}_{ds} - k \frac{\omega_{\mathrm{s}}}{\omega_{\mathrm{b}}} x'_{\mathrm{r}} \lambda^{\mathrm{s}}_{qs}$$

(524-525)

where the definition used for the slip speed is: $\omega_s = \omega_e - \omega_r$. The electromagnetic torque expression has to be re-written:

$$T_e = \frac{P}{2} \cdot \frac{1}{\omega_b} \left(\frac{1}{k} \lambda_{ds}^s i_{qs}^s - k \lambda_{qs}^s i_{ds}^s \right) \tag{526}$$

If we make the notation: $i_{qs}^s = k^2 i_{qs1}^s$ the above torque relation may be expressed as for the symmetrical induction machine:

$$T_e = \frac{P}{2} \cdot \frac{k}{\omega_b} \left(\lambda_{ds}^s i_{qs1}^s - \lambda_{qs}^s i_{ds}^s \right) \tag{527}$$

If the special reference frame is fixed to the stator flux linkage vector, the q-component of this flux vector is defined equal to zero:

$$\lambda_{qs}^s = 0$$
$$\lambda_{ds}^s = \lambda_s^s \tag{528}$$

From stator flux linkage equations, the q-current components are given by:

$$i_{qs1}^s = i_{qs1}^s$$
$$i_{qr}^{\prime s} = -\frac{k^2 x_s}{x_m} i_{qs1}^s \tag{529}$$

The electromagnetic torque relation and slip speed can be derived in stator field orientation control as:

$$T_e = \frac{P}{2} \cdot \frac{k}{\omega_b} \lambda_{ds}^s i_{qs1}^s$$
$$\omega_s = \frac{k^2 \left(\omega_b r_r' x_s + D \cdot p \right) i_{qs1}^s}{\frac{1}{k} x_r' \lambda_{ds}^s - k \cdot D \cdot i_{ds}^s} \tag{530-531}$$

The second dynamic equation of the machine, shows that there is a coupling between the stator current components. Consequently, any change in the torque producing component i_{qs1}^s without changing i_{ds}^s accordingly will cause a transient in the stator flux. A decoupler is necessary to overcome this disadvantage. Therefore the command current of the d-axis component of the stator current can be calculated as follows:

$$i_{ds}^s = \left(K_p + \frac{K_i}{p} \right) \Delta \lambda_{ds}^s + i_{dq}^s$$
$$i_{dq}^s = \frac{k \omega_s D \cdot i_{qs1}^s}{\omega_b r_r' x_s + D \cdot p} \tag{532-533}$$

where K_p and K_i are proportional, respectively integral coefficients of the flux controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance of a stator flux oriented system

Letting the derivative operator $p = 0$, one can obtain the steady-state voltage equations of the induction machine. After several manipulation of the system equations, the d current components are given by:

$$i_{ds}^{s} = \frac{kD}{r'_{r} x_{s}} \cdot \frac{\omega_{s}}{\omega_{b}} i_{qs1}^{s} + \frac{\lambda_{ds}^{s}}{k^{2} x_{s}}$$

$$i_{dr}^{'s} = \frac{\lambda_{ds}^{s}}{k^{2} x_{m}} - \frac{x_{s}}{x_{m}} i_{ds}^{s}$$

(534-535)

it yields the slip speed equation:

$$\left(\frac{\omega_{s}}{\omega_{b}}\right)^{2} i_{qs1}^{s} \frac{k^{2} D^{2}}{r'_{r} x_{s}} - \frac{\omega_{s}}{\omega_{b}} \cdot \frac{x_{m}^{2}}{kx_{s}} \lambda_{s}^{s} + k^{2} r'_{r} x \, i_{qs1}^{s} = 0$$

(536)

The solutions of the above equations have to be real, for a given stator flux linkage. This means that the determinant of the second order equation satisfy the condition:

$$\Delta = \left(\frac{x_{m}^{2} \lambda_{s}^{s}}{kx_{s}}\right)^{2} - 4k^{4} D^{2} \left(i_{qs1}^{s}\right)^{2} \geq 0$$

The maximum values for the q component of the stator current, the slip speed and the electromagnetic torque (pull-out torque) are:

$$\left(i_{qs1}^{s}\right)_{max} = \frac{x_{m}^{2} \lambda_{s}^{s}}{2k^{2} D x_{s}}$$

$$\left(\omega_{s}\right)_{max} = \frac{r'_{r} x_{s}}{kD}$$

(537-539)

$$\left(T_{e}\right)_{max} = \frac{P}{2k^{2}} \cdot \frac{x_{m}^{2} \left(\lambda_{s}^{s}\right)^{2}}{2\omega_{b} D x_{s}}$$

The problems related to static stability of the single-phase induction machine when this vector control strategy is applied, are similar to the polyphase machine case. It has to pointed out that the critical values for current q-component, slip angular velocity and electromagnetic torque depend also on the turns ratio k of the unsymmetrical stator windings. If we detail this consideration, it can be observed that having a motor equipped with a main stator winding with fixed parameters (resistance, reactance) the pull out torque and critical rotor speed are such that (Popescu -2000):

- A maximum torque value and minimum rotor speed are obtained if the turns ratio $k < 1$, which is the case of the usual split-phase motors configuration;
- The medium values are obtained if the turns ratio $k = 1$, which is the case of the symmetrical motors configuration;
- A minimum torque value and maximum rotor speed are obtained if the turns ratio $k > 1$, which is the case of the capacitor run motors configuration;

Some important conclusions can be drawn for this vector control strategy, besides those ones valid for the polyphase motor:

- The estimated equivalent value for the torque component of the stator current has to be corrected by dividing to the square of the turns ratio value;
- There is no need for changing the number of co-ordinates from the real machine system to the control system.

3.8.2. Rotor flux orientation (RFO)

For this vector control strategy, the set of state-space variables formed by rotor flux linkage and stator current space vectors $x = \left[\lambda'_{qr}, \lambda'_{dr}, i_{qs}, i_{ds}\right]^{T}$ is selected. The mathematical model is given in the chapter dedicated to d-q models of the single-phase induction machine.

The stator flux linkage and rotor current space vectors components expressed as functions in terms of state-space variables are:

$$\lambda_{qs}^{r} = \frac{1}{x'_{r}}\left(x_{m}\lambda''_{qr} + D\cdot i_{qs}^{r}\right)$$

$$\lambda_{ds}^{r} = \frac{1}{x'_{r}}\left(x_{m}\lambda''_{dr} + k^{2}D\cdot i_{ds}^{r}\right)$$

(540-543)

$$i''_{qr} = \frac{1}{x'_{r}}\left(\lambda''_{qr} - x_{m}i_{qs}^{r}\right)$$

$$i''_{dr} = \frac{1}{x'_{r}}\left(\frac{\lambda''_{dr}}{k^{2}} - x_{m}i_{ds}^{r}\right)$$

The rotor voltage equations re-written in terms of the state-space variables become:

$$u''_{qr} = 0 = -x_{m}\frac{r'_{r}}{x'_{r}}i_{qs}^{r} + \left(\frac{r'_{r}}{x'_{r}} + \frac{p}{\omega_{b}}\right)\lambda''_{qr} + \frac{1}{k}\cdot\frac{\omega_{s}}{\omega_{b}}\lambda''_{dr}$$

(544-545)

$$u''_{dr} = 0 = -k^{2}x_{m}\frac{r'_{r}}{x'_{r}}i_{ds}^{r} + \left(\frac{r'_{r}}{x'_{r}} + \frac{p}{\omega_{b}}\right)\lambda''_{dr} - k\frac{\omega_{s}}{\omega_{b}}\lambda''_{qr}$$

The electromagnetic torque expression has to be re-written:

$$T_{e} = \frac{P}{2}\cdot\frac{x_{m}}{x'_{r}\,\omega_{b}}\left(\frac{1}{k}\lambda''_{dr}\,i_{qs}^{r} - k\lambda''_{qr}\,i_{ds}^{r}\right)$$

(546)

If we make the notation: $i_{qs}^{r} = k^{2}i_{qs1}^{r}$ the above torque relation may be expressed as for the symmetrical induction machine:

$$T_{e} = \frac{P}{2}\cdot\frac{x_{m}}{x'_{r}}\frac{k}{\omega_{b}}\left(\lambda''_{dr}\,i_{qs1}^{r} - \lambda''_{qr}\,i_{ds}^{s}\right)$$

(547)

If the special reference frame is fixed to the rotor flux linkage vector, the q-component of this flux vector is defined equal to zero:

$$\lambda''_{qr} = 0$$

$$\lambda''_{dr} = \lambda''_{r}$$

(548)

From rotor flux linkage equations, the q-current components are given by:

$$i_{qs1}^{r} = i_{qs1}^{r}$$

$$i''_{qr} = -\frac{k^{2}x_{m}}{x'_{r}}\cdot i_{qs1}^{r}$$

(549)

The flux producing component of the stator current is determined as follows:

$$i_{ds}^{r} = \frac{\left(1 + \frac{x'_{r}}{r'_{r}}\cdot\frac{p}{\omega_{b}}\right)\lambda''_{dr}}{k^{2}x_{m}}$$

(550)

The above relation shows that there is no need of a current decoupler in rotor field orientation scheme. Both stator current components (torque and flux producing) can be controlled independently.

Steady-state perfomance of a rotor flux oriented system

Letting the derivative operator $p = 0$, one can obtain the steady-state voltage equations of the induction machine. After several manipulation of the system equations, the d current components are given by:

$$i_{ds}^r = \frac{\lambda_{dr}^{'r}}{k^2 x_m} = \frac{\omega_b}{\omega_s} \cdot \frac{k \cdot r'_r i_{qs1}^r}{x'_r}$$

$$(551)$$

$$i'^r_{dr} = 0$$

It yields the slip speed equation:

$$\frac{\omega_s \lambda_{dr}^{'r}}{k^2} - k \cdot \frac{x_m}{x'_r} \cdot \omega_b r'_r i_{qs1}^r = 0$$

$$(552)$$

which for a given rotor flux has always real solution. Thus the resulting current controlled slip speed and the electromagnetic torque are:

$$\omega_s = \frac{k^3 x_m r'_r \omega_b i_{qs1}^r}{x'_r \lambda_{dr}^{'r}} = \frac{k \cdot x_m r'_r \omega_b i_{qs}^r}{x'_r \lambda_{dr}^{'r}}$$

$$(553\text{-}554)$$

$$T_e = \frac{P}{2} \cdot k \cdot \frac{x_m}{x'_r \omega_b} \lambda_{dr}^{'r} i_{qs1}^r = \frac{P}{2} \cdot \left(\frac{x_m}{x'_r}\right)^2 \left(i_{qs1}^r\right)^2 \cdot \frac{k^4 r'_r}{\omega_s} = \frac{P}{2} \cdot \left(\frac{x_m}{x'_r}\right)^2 \left(i_{qs}^r\right)^2 \cdot \frac{k^2 r'_r}{\omega_s}$$

By comparison with the three-phase induction motor, for the rotor field orientation control strategy, three differences have been highlighted by Popescu and Navrapescu (2000):

- The stady-state values for the slip angular velocity and torque are proportional with the turns ratio value, respectively the square of turns ratio k which determines the option for $k > 1$, as the torque response is the determinant factor in an electrical drive system;
- The estimated equivalent value for the torque component of the stator current has to be corrected by dividing to the square of the turns ratio value;
- There is no need for changing the number of co-ordinates from the real machine system to the control system;
- The problem of the stator windings asymmetrical configuration is overcome by using the value of turns ratio k in computing the estimated value for the torque and flux producing components of the stator current.

3.8.3. Air-gap flux field orientation (AFO)

For this vector control strategy, the set of state-space variables formed by air-gap flux linkage and stator current space vectors $x = \left[\lambda_{qm}, \lambda_{dm}, i_{qs}, i_{ds}\right]^T$ is selected. The mathematical model is given in the chapter dedicated to d-q models of the single-phase induction machine.

The stator and rotor flux linkages and rotor current space vectors components expressed as functions in terms of state-space variables are:

$$\lambda_{qs}^m = \lambda_{qm}^m + \left(x_s - x_m\right) i_{qs}^m$$

$$\lambda_{ds}^m = \lambda_{dm}^m + k^2 \left(x_s - x_m\right) i_{ds}^m$$

$$\lambda_{qr}^{'m} = \frac{x'_r}{x_m} \lambda_{qm}^m - \left(x'_r - x_m\right) i_{qs}^m$$

$$(555\text{-}558)$$

$$\lambda_{dr}^{'m} = \frac{x'_r}{x_m} \lambda_{qm}^m - k^2 \left(x'_r - x_m\right) i_{ds}^m$$

$$i_{qr}^{'r} = \frac{1}{x_m} \lambda_{qm}^m - i_{qs}^m$$

$$(559\text{-}560)$$

$$i_{dr}^{'r} = \frac{1}{k^2 x_m} \lambda_{dm}^m - i_{ds}^m$$

The voltage equations of the machine expressed in state-variables terms, are readily deductible now:

$$u'^{m}_{qr} = 0 = -\left(r'_r + (x'_r - x_m)\frac{p}{\omega_b}\right)i^{m}_{qs} + \left(\frac{r'_r}{x_m} + \frac{x'_r}{x_m}\cdot\frac{p}{\omega_b}\right)\lambda^{m}_{qm} + \frac{\omega_s}{\omega_b}\left(\frac{1}{k}\cdot\frac{x'_r}{x_m}\lambda^{m}_{dm} - k\cdot(x'_r - x_m)i^{m}_{ds}\right)$$

$$u'^{m}_{dr} = 0 = -k^2\left(r'_r + (x'_r - x_m)\frac{p}{\omega_b}\right)i^{m}_{ds} + \left(\frac{r'_r}{x_m} + \frac{x'_r}{x_m}\cdot\frac{p}{\omega_b}\right)\lambda^{m}_{dm} - k\frac{\omega_s}{\omega_b}\left(\frac{x'_r}{x_m}\lambda^{m}_{qm} - (x'_r - x_m)i^{m}_{qs}\right)$$

(561-562)

The electromagnetic torque expression has to be re-written:

$$T_e = \frac{P}{2}\cdot\frac{1}{\omega_b}\left(\frac{1}{k}\lambda^{m}_{dm}i^{m}_{qs} - k\,\lambda^{m}_{qm}i^{m}_{ds}\right)$$

(563)

If we make the notation: $i^{m}_{qs} = k^2 i^{m}_{qs1}$ the above torque relation may be expressed as for the symmetrical induction machine:

$$T_e = \frac{P}{2}\cdot\frac{k}{\omega_b}\left(\lambda^{m}_{dm}i^{m}_{qs1} - \lambda^{m}_{qm}i^{m}_{ds}\right)$$

(564)

If the special reference frame is fixed to the air-gap flux linkage vector, the q-component of this flux vector is defined equal to zero:

$$\lambda^{m}_{qm} = 0$$

$$\lambda^{m}_{dm} = \lambda^{m}_{m}$$

(565)

From the air-gap flux linkage equations, the q-current components are given by:

$$i^{m}_{qs1} = i^{m}_{qs1}$$

$$i'^{m}_{qr} = -k^2\cdot i^{m}_{qs1}$$

(566)

The flux producing component of the stator current is determined as follows:

$$k^2\left(r'_r + (x'_r - x_m)\frac{p}{\omega_b}\right)i^{m}_{ds} = \left(\frac{r'_r}{x_m} + \frac{x'_r}{x_m}\cdot\frac{p}{\omega_b}\right)\lambda^{m}_{dm} + k^3(x'_r - x_m)\frac{\omega_s}{\omega_b}i^{m}_{qs1}$$

(567)

When an air-gap field orientation control is employed, it is necessary to decouple the stator current components, in order to achieve a linear control. For this reason, the command of current of the d-axis component is computed as follows:

$$i^{m}_{ds} = \left(K_p + \frac{K_i}{p}\right)\Delta\lambda^{m}_{dm} + i^{m}_{dq}$$

$$i^{m}_{dq} = \frac{k\cdot\omega_s(x'_r - x_m)\cdot i^{m}_{qs1}}{\omega_b r'_r + (x'_r - x_m)p}$$

(568-569)

The electromagnetic torque relation and slip speed can be derived in air-gap field orientation control as:

$$T_e = \frac{P}{2}\cdot\frac{k}{\omega_b}\lambda^{m}_{dm}i^{m}_{qs}$$

$$\omega_s = \frac{\left[\omega_b r'_r + (x'_r - x_m)p\right]\cdot k^2 i^{m}_{qs}}{\dfrac{x'_r}{x_m}\cdot\dfrac{1}{k}\lambda^{m}_{dm} - k\cdot(x'_r - x_m)\cdot i^{m}_{ds}}$$

(570-571)

Steady-state perfomance of a air-gap flux oriented system

Letting the derivative operator $p = 0$, the steady-state voltage equations of the induction machine are readily determined. After several manipulation of the system equations, the d current components are given by:

$$i_{ds}^m = \frac{k}{x_m}\left(r'_r \cdot \frac{\omega_b}{\omega_s} i_{qs1}^m + \frac{x'_r}{r'_r}(x'_r - x_m) \cdot \frac{\omega_s}{\omega_b} i_{qs1}^m \right)$$

$$i'^m_{dr} = \frac{k(x_m - x'_r)i_{qs1}^m}{r'_r} \cdot \frac{\omega_s}{\omega_b}$$

(572)

It yields the slip speed equation:

$$\left(\frac{\omega_s}{\omega_b}\right)^2 \cdot \frac{k(x'_r - x_m)^2}{r'_r} \cdot i_{qs1}^m - \frac{\omega_s}{\omega_b} \cdot \frac{\lambda_{dm}^m}{k^2} + k \cdot r'_r \; i_{qs1}^m = 0$$

(573)

The solutions of the above equations have to be real, for a given stator flux linkage. This means that the determinant of the second order equation satisfy the condition:

$$\Delta = \left(\frac{\lambda_{dm}^m}{k^2}\right)^2 - 4k^2 \left(x'_r - x_m\right)^2 \cdot \left(i_{qs1}^m\right)^2 \geq 0$$

(574)

The maximum values for the q component of the stator current, the slip speed and the electromagnetic torque (pull-out torque) are:

$$\left(i_{qs1}^m\right)_{max} = \frac{\lambda_{dm}^m}{2k^3(x'_r - x_m)}$$

$$\left(\omega_s\right)_{max} = \frac{r'_r}{x'_r - x_m}$$

(575-577)

$$\left(T_e\right)_{max} = \frac{P}{2} \cdot \frac{1}{2\omega_b} \cdot \frac{\left(\lambda_{dm}^m\right)^2}{k^2\left(x'_r - x_m\right)}$$

For the air-gap field orientation control strategy, there have to be pointed out several important features:

- The turns ratio k determines the effects on the critical value of the electromagnetic torque of the single-phase machine: lower torque for $k > 1$, and higher torque for $k < 1$, when the main stator winding parameters are kept constant;
- The pull-out value of the slip speed does not depend on the unsymetrical configuration of the stator windings, and it is expressed with an identical relation to that deduced for the three-phase induction machine;
- The estimated equivalent value for the torque component of the stator current has to be corrected by dividing to the square of the turns ratio value;
- There is no need for changing the number of co-ordinates from the real machine system to the control system;
- The problem of the stator windings asymmetrical configuration is overcome by using the value of turns ratio k in computing the estimated value for the torque and flux producing components of the stator current.

3.8.4. Stator current orientation control (SCO)

For a complete comparison to the vector control strategies applied to three-phase induction motor, the current orientation control shemes have to be analysed as well. One option is if the set of state-space variables is identical with the stator flux field orientation control (SFO): the stator flux

and current space vectors $x = \left[\lambda_{qs}, \lambda_{ds}, i_{qs}, i_{ds} \right]^{\mathrm{T}}$. The mathematical model is identical to that used in SFO case.

The rotor based variables expressed in state-space variables terms are:

$$i'^{sc}_{qr} = \frac{1}{x_m}\left(\lambda^{sc}_{qs} - x_s i^{sc}_{qs} \right)$$

$$i'^{sc}_{dr} = \frac{1}{k^2 x_m}\left(\lambda^{sc}_{ds} - k^2 x_s i^{sc}_{ds} \right)$$

$$\lambda'^{sc}_{qr} = \frac{1}{x_m}\left(x'_r \lambda^{sc}_{qs} - D \cdot i^{sc}_{qs} \right) \tag{578-581}$$

$$\lambda'^{sc}_{dr} = \frac{1}{x_m}\left(x'_r \lambda^{sc}_{ds} - k^2 D \cdot i^{sc}_{ds} \right)$$

To obtain the machine equations in the synchronuous stator current reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the stator current to be zero. The stator voltages equations remain unchanged. If a cage rotor is considered, the resultant equations are as follows:

$$\left(r'_r x_s + D \frac{p}{\omega_b} \right) i^{sc}_{qs} - \frac{\omega_s}{\omega_b}\left(\frac{1}{k} x'_r \lambda^{sc}_{ds} - k \cdot D \cdot i^{sc}_{ds} \right) - \left(r'_r + x'_r \frac{p}{\omega_b} \right) \lambda^{sc}_{qs} = 0$$

$$k^2 \left(r'_r x_s + D \frac{p}{\omega_b} \right) i^{sc}_{ds} - k \frac{\omega_s}{\omega_b} D \cdot i^{sc}_{qs} = \left(r'_r + x'_r \frac{p}{\omega_b} \right) \lambda^{sc}_{ds} - k \frac{\omega_s}{\omega_b} x'_r \lambda^{sc}_{qs} \tag{582-583}$$

The electromagnetic torque expression has to be re-written:

$$T_e = \frac{P}{2} \cdot \frac{1}{\omega_b}\left(\frac{1}{k} \lambda^{sc}_{ds} i^{sc}_{qs} - k \lambda^{sc}_{qs} i^{sc}_{ds} \right) \tag{584}$$

If we make the notation: $\lambda^{sc}_{qs} = \frac{1}{k^2} \lambda^{sc}_{qs1}$ the above torque relation may be expressed as for the symmetrical induction machine:

$$T_e = \frac{P}{2} \cdot \frac{1}{k\omega_b}\left(\lambda^{sc}_{ds} i^{sc}_{qs} - \lambda^{sc}_{qs1} i^{sc}_{ds} \right) \tag{585}$$

If the special reference frame is fixed to the stator current space vector, the q-component of this current vector is defined equal to zero:

$$i^{sc}_{qs} = 0$$

$$i^s_s = i^{sc}_{ds} \tag{586}$$

Different from the SFO case, we have to express the q-flux components, by considering the flux linkages equations:

$$\lambda^{sc}_{qs1} = \lambda^{sc}_{qs1}$$

$$\lambda'^{sc}_{qr} = \frac{x'_r}{k^2 x_m} \lambda^{sc}_{qs1} \tag{587}$$

The electromagnetic torque relation and slip speed can be derived in stator current orientation control as:

$$T_e = -\frac{P}{2} \cdot \frac{1}{k\omega_b} \lambda^{sc}_{qs1} i^{sc}_{ds}$$

$$\omega_s = \frac{\left(\omega_b r'_r x_s + x'_r \cdot p \right) \lambda^{sc}_{qs1}}{k \left(k^2 D \cdot i^s_{ds} - x'_r \lambda^{sc}_{ds} \right)} \tag{588-589}$$

The relation between the d-axis component of the stator current and stator flux linkage space vector components can be deduced as follows:

$$k^2 \left(r'_r \, x_s + D \frac{p}{\omega_b} \right) i_{ds}^{sc} - \left(r'_r + x'_r \frac{p}{\omega_b} \right) \lambda_{ds}^{sc} + \frac{1}{k} \cdot \frac{\omega_s}{\omega_b} x'_r \lambda_{qs1}^{sc} = 0 \tag{590}$$

Similarly to the SFO case, there is a coupling, but between the stator flux linkage components used as control variables. Consequently, any change in the torque producing component λ_{ds}^{sc} without changing λ_{qs}^{sc} accordingly, will cause a transient in the stator flux. A decoupler is necessary to overcome this disadvantage:

$$\lambda_{ds}^{sc} = \left(K_p + \frac{K_i}{p} \right) \Delta i_{ds}^{sc} + \lambda_{dq}^{sc}$$

$$\lambda_{dq}^{sc} = \frac{\dfrac{1}{k} \cdot \dfrac{\omega_s}{\omega_b} x'_r \lambda_{qs1}^{sc}}{\left(r'_r + x'_r \dfrac{p}{\omega_b} \right)} \tag{591-592}$$

where K_p and K_i are proportional, respectively integral coefficients of the current controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance of a stator current oriented system

Letting the derivative operator $p = 0$, one can obtain the steady-state voltage equations of the induction machine:

$$\lambda_{ds}^{sc} = -\lambda_{qs1}^{sc} \cdot \frac{x'_r}{kr'_r} \cdot \frac{D}{x_m^2} \cdot \frac{\omega_s}{\omega_b} - \lambda_{qs1}^{sc} \cdot \frac{x_s r'_r}{kx_m^2} \cdot \frac{\omega_b}{\omega_s}$$

$$\lambda_{dr}^{sc} = -\frac{r'_r}{kx_m} \cdot \frac{\omega_b}{\omega_s} \lambda_{qs1}^{sc} \tag{593}$$

it yields the slip speed equation:

$$\left(\frac{\omega_s}{\omega_b} \right)^2 \left(\frac{x'_r}{x_m} \right)^2 \cdot \frac{1}{k \cdot r'_r} \lambda_{qs1}^{sc} + \left(\frac{\omega_s}{\omega_b} \right) \cdot k^2 i_{ds}^{sc} + \frac{r'_r}{k \cdot x_m^2} \lambda_{qs1}^{sc} = 0 \tag{594}$$

The solutions of the above equations have to be real, for a given stator current. This means that the determinant of the second order equation satisfy the condition:

$$\Delta = \left(k^2 i_{ds}^{sc} \right)^2 - 4 \left(\frac{x'_r}{x_m} \right)^2 \left(\frac{\lambda_{qs}^{sc}}{k^2 x_m^2} \right)^2 \geq 0 \tag{595}$$

The maximum values for the q component of the stator flux linkage, the slip speed and the electromagnetic torque (pull-out torque) are:

$$\left(\lambda_{qs1}^{sc} \right)_{max} = \frac{k^2 x_m^2 i_{ds}^{sc}}{2 x'_r}$$

$$\left(\omega_s \right)_{max} = \frac{k \cdot r'_r}{x'_r} \tag{596-598}$$

$$\left(T_e \right)_{max} = \frac{P}{2} \cdot \frac{k \cdot x_m^2 \left(i_{ds}^{sc} \right)^2}{2 \omega_b x'_r}$$

The analysis of this vector control strategy leads to the same conclusions as for the three-phase induction machine case, but with the differences imposed by the asymmetrical motor configuration and the number of phases for the supply voltage:

- The turns ratio k determines different effects on the critical values of the rotor speed and electromagnetic torque response of the single-phase machine: the electromagnetic torque critical values reaches a maximum for $k > 1$, while the rotor speed reaches a maximum for $k < 1$, when the main stator winding parameters are kept constant;
- The estimated equivalent value for the torque component of the stator flux has to be corrected by multypling with the square of the turns ratio value k;
- There is no need for changing the number of co-ordinates from the real machine system to the control system;

3.8.5. Rotor current orientation control (RCO)

If the synchronuous reference frame is linked to the rotor current space vector it results another unconventional vector control strategy. The rotor current orientation control strategy (RCO) is analysed for a cage rotor induction motor. There are two options in selecting the set of state-space variables, if we consider the criteria of direct measurable quantities:

I. The state-space variables are the stator flux linkage and rotor current space vectors: $x = \left[\lambda_{qs}, \lambda_{ds}, i'_{qr}, i'_{dr}\right]^{\mathrm{T}}$. When the stator flux and and the rotor current are selected as state-space variables, one can derive the rotor flux and stator current functions in terms of state variables as follows:

$$i_{qs}^{rc} = \frac{1}{x_s}\left(\lambda_{qs}^{rc} - x_m i'^{rc}_{qr}\right)$$

$$i_{ds}^{rc} = \frac{1}{x_s}\left(\frac{\lambda_{ds}^{rc}}{k^2} - x_m i'^{rc}_{dr}\right)$$

$$\lambda'^{rc}_{qr} = \frac{x_m}{x_s}\lambda_{qs}^{rc} + \frac{D}{x_s}i'^{rc}_{qr}$$ (599-602)

$$\lambda'^{rc}_{dr} = \frac{x_m}{x_s}\lambda_{ds}^{rc} + \frac{k^2 D}{x_s}i'^{rc}_{dr}$$

To obtain the machine equations in the synchronuous rotor current reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the rotor current to be zero:

$$0 = \left(r'_r + \frac{D}{x_s}\cdot\frac{p}{\omega_b}\right)i^{rc}_{qr} + \frac{kD}{x_s}\cdot\frac{\omega_s}{\omega_b}i^{rc}_{dr} + \frac{x_m}{x_s}\cdot\frac{p}{\omega_b}\lambda_{qs}^{rc} + \frac{1}{k}\cdot\frac{x_m}{x_s}\cdot\frac{\omega_s}{\omega_b}\lambda_{ds}^{rc}$$

$$0 = k^2\left(r'_r + \frac{D}{x_s}\cdot\frac{p}{\omega_b}\right)i^{rc}_{dr} - \frac{kD}{x_s}\cdot\frac{\omega_s}{\omega_b}i'^{rc}_{qr} + \frac{x_m}{x_s}\cdot\frac{p}{\omega_b}\lambda_{ds}^{rc} - \frac{kx_m}{x_s}\cdot\frac{\omega_s}{\omega_b}\lambda_{qs}^{rc}$$ (603-604)

The electromagnetic torque expression has to be re-written:

$$T_e = \frac{P}{2}\cdot\frac{x_m}{\omega_b x_s}\left(k\lambda_{qs}^{rc}i'^{rc}_{dr} - \frac{1}{k}\lambda_{ds}^{rc}i'^{rc}_{qr}\right)$$ (605)

If we make the notation: $\lambda_{qs}^{rc} = \frac{1}{k^2}\lambda_{qs1}^{rc}$ the above torque relation may be expressed as for the symmetrical induction machine:

$$T_e = \frac{P}{2}\cdot\frac{x_m}{k\omega_b x_s}\left(\lambda_{qs1}^{rc}i'^{rc}_{dr} - \lambda_{ds}^{rc}i'^{rc}_{qr}\right)$$ (606)

If the special reference frame is fixed to the rotor current space vector, the *q*-component of this current vector is defined equal to zero:

$$i'^{\text{rc}}_{qr} = 0$$

$$i'^{\text{rc}}_{dr} = i'^{\text{rc}}_{r} \tag{607}$$

Different from the RFO case, we have to express the *q*-flux components, by considering the flux linkages equations:

$$\lambda^{\text{rc}}_{qs1} = \lambda^{\text{rc}}_{qs1}$$

$$\lambda'^{\text{rc}}_{qr} = \frac{x_s}{k^2 x_m} \lambda^{\text{sc}}_{qs1} \tag{608}$$

The electromagnetic torque relation and slip speed can be derived in stator current orientation control as:

$$T_e = \frac{P}{2} \cdot \frac{x_m}{k x_s \omega_b} \lambda^{\text{rc}}_{qs1} i'^{\text{rc}}_{dr}$$

$$\omega_s = \frac{-\dfrac{x_m}{k^2} \cdot p \lambda^{\text{rc}}_{qs1}}{k \cdot D \cdot i'^{\text{rc}}_{dr} + \dfrac{1}{k} x_m \lambda^{\text{rc}}_{ds}} \tag{609-610}$$

A relation between the d-axis component of the rotor current and stator flux linkage space vector components can be deduced:

$$0 = k^2 \left(r'_r + \frac{D}{x_s} \cdot \frac{p}{\omega_b} \right) i'^{\text{rc}}_{dr} + \frac{x_m}{x_s} \cdot \frac{p}{\omega_b} \lambda^{\text{rc}}_{ds} - \frac{x_m}{k x_s} \cdot \frac{\omega_s}{\omega_b} \lambda^{\text{rc}}_{qs1} \tag{611}$$

Due to the coupling between the stator flux linkage components, any change in the torque producing component λ^{rc}_{ds} without changing λ^{rc}_{qs} accordingly, will cause a trensient in the stator flux. To overcome this disadvantage, the command current of the d-axis component of the stator current is expressed as follows:

$$\lambda^{\text{rc}}_{ds} = \left(K_p + \frac{K_i}{p} \right) \Delta i'^{\text{rc}}_{dr} + \lambda^{\text{rc}}_{dq}$$

$$\lambda^{\text{rc}}_{dq} = \frac{1}{p} \cdot \omega_s \cdot \frac{\lambda^{\text{sc}}_{qs1}}{k} \tag{612-613}$$

where K_p and K_i are proportional, respectively integral coefficients of the current controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance

After several manipulation of the system equations if we let the derivative operator $p = 0$, the *d* current components are obtained from the steady-state voltage equations of the induction machine:

$$\lambda^{\text{rc}}_{ds} = -\frac{D}{k x_s r'_r} \cdot \frac{\omega_s}{\omega_b} \lambda^{\text{rc}}_{qs1} \tag{614}$$

$$\lambda'^{\text{rc}}_{dr} = 0$$

It yields the slip angular velocity equation:

$$i'^{\text{rc}}_{dr} \cdot \frac{k^2 D}{x_s} - \frac{x_m D}{k x_s^2 r'_r} \cdot \frac{\omega_s}{\omega_b} \lambda^{\text{rc}}_{qs1} = 0 \tag{615}$$

which for a given rotor flux has always real solution. Thus the slip speed and the electromagnetic torque are:

$$\omega_s = \frac{k^3 x_s r'_r \omega_b i''^{rc}_{dr}}{x_m \lambda^{rc}_{qs1}} = \frac{k x_s r'_r \omega_b i''^{rc}_{dr}}{x_m \lambda^{rc}_{qs}}$$

$$T_e = \frac{P}{2} \cdot \frac{x_m}{k x_s \omega_b} \lambda^{rc}_{qs1} i''^{rc}_{dr} = \frac{P}{2} \cdot \frac{k x_m}{x_s \omega_b} \lambda^{rc}_{qs} i''^{rc}_{dr}$$

(616-617)

The rotor current oriented (RCO) vector control with mixed flux and current state-space variables differs from the variant applied to the three-phase induction machine by the following:

- The transformation between the reference frames (stationary to synchronous and vice-versa) is easier as there is no need of changing the number of variables;
- The asymmetry of the stator windings is modeled only by using a supplementary parameter, the turns ratio k;
- In steady-state operation, the torque and slip speed response are influenced by the unsymmetrical configuration of the motor: higher torque and slip speed for $k < 1$.

II. The state-space variables are stator and rotor currents $x = \begin{bmatrix} i_{qs}, i_{ds}, i_{qr}, i_{dr} \end{bmatrix}^T$. When the stator and the rotor current are selected as state-space variables, one can derive the stator and rotor flux as functions in terms of state variables from the classical flux linkage equations.

To obtain the machine equations in the synchronuous rotor current reference frame, one has to eliminate the rotor flux from the rotor voltage equations and then force the q component of the rotor current to be zero. The stator voltages equations are re-written also. The resultant equations have the following form:

$$0 = x_m \frac{p}{\omega_b} i^{rc}_{qs} + \left(r'_r + x'_r \frac{p}{\omega_b} \right) i''^{rc}_{qr} + k x_m \frac{\omega_s}{\omega_b} i^{rc}_{ds} + k x'_r \frac{\omega_s}{\omega_b} i''^{rc}_{dr}$$

$$0 = -k x_m \frac{\omega_s}{\omega_b} i^{rc}_{qs} - k x'_r \frac{\omega_s}{\omega_b} i''^{rc}_{qr} + k^2 x_m \frac{p}{\omega_b} i^{rc}_{ds} + k^2 \left(r'_r + x'_r \frac{p}{\omega_b} \right) i''^{rc}_{dr}$$

(618-619)

The electromagnetic torque expression has the expression:

$$T_e = \frac{P}{2} \cdot \frac{k x_m}{\omega_b} \left(i^{rc}_{qs} i''^{rc}_{dr} - i^{rc}_{ds} i''^{rc}_{qr} \right)$$

(620)

It can be observed that the above relation is readily available for implementing a vector control strategy.

When linking the special synchronuous reference frame to the rotor current space vector, the q-component of this current vector is defined equal to zero:

$$i''^{rc}_{qr} = 0$$

$$i''^{rc}_{dr} = i''^{rc}_r$$

(621)

Different from the (*I*) case of RCO control strategy, for the (*II*) case we have to express the stator flux linkage q-components, by considering the flux linkages equations:

$$\lambda^{rc}_{qs} = x_s i^{rc}_{qs}$$

$$\lambda''^{rc}_{qr} = x_m i^{sc}_{qs} = \frac{x_m}{x_s} \lambda^{rc}_{qs}$$

(622)

The electromagnetic torque relation and slip speed can be derived in this type of rotor current orientation control as:

$$T_{\mathrm{e}} = \frac{P}{2} \cdot \frac{kx_{\mathrm{m}}}{\omega_{\mathrm{b}}} \cdot i_{\mathrm{qs}}^{\mathrm{rc}} i_{\mathrm{dr}}^{\prime \mathrm{rc}}$$

$$\omega_{\mathrm{s}} = -\frac{x_{\mathrm{m}} \cdot p i_{\mathrm{qs}}^{\mathrm{rc}}}{k \left(x_{\mathrm{m}} i_{\mathrm{ds}}^{\mathrm{rc}} + x_{\mathrm{r}}^{\prime} \, i_{\mathrm{dr}}^{\prime \mathrm{rc}} \right)}$$

<div align="right">(623-624)</div>

From the second dynamic equation of the machine, a relation between the *d*-axis component of the rotor current and stator current space vector components can be expressed as:

$$k^2 x_{\mathrm{m}} \frac{p}{\omega_{\mathrm{b}}} i_{\mathrm{ds}}^{\mathrm{rc}} = kx_{\mathrm{m}} \frac{\omega_{\mathrm{s}}}{\omega_{\mathrm{b}}} i_{\mathrm{qs}}^{\mathrm{rc}} - k^2 \left(r_{\mathrm{r}}^{\prime} + x_{\mathrm{r}}^{\prime} \frac{p}{\omega_{\mathrm{b}}} \right) i_{\mathrm{dr}}^{\mathrm{rc}}$$

<div align="right">(625)</div>

A coupling between the *d*-axis and *q*-axis stator current components appears, and consequently, any change in the torque producing component $i_{\mathrm{ds}}^{\mathrm{rc}}$ without changing $i_{\mathrm{qs}}^{\mathrm{rc}}$ accordingly, will cause a transient in the stator flux. To overcome this disadvantage the command current of the *d*-axis component of the stator current has to be expressed in the following form:

$$i_{\mathrm{ds}}^{\mathrm{rc}} = \left(K_{\mathrm{p}} + \frac{K_{\mathrm{i}}}{p} \right) \Delta i_{\mathrm{dr}}^{\prime \mathrm{rc}} + i_{\mathrm{dq}}^{\mathrm{rc}}$$

$$i_{\mathrm{dq}}^{\mathrm{rc}} = \frac{1}{p} \cdot \omega_{\mathrm{s}} \cdot \frac{i_{\mathrm{qs}}^{\mathrm{sc}}}{k}$$

<div align="right">(626-627)</div>

where K_{p} and K_{i} are proportional, respectively integral coefficients of the current controller. This controller can be PI type or soft computing technique type (fuzzy, neural-network).

Steady-state perfomance

By letting the derivative operator $p = 0$, the *d* flux components are obtained from the steady-state voltage equations of the induction machine:

$$\lambda_{\mathrm{ds}}^{\mathrm{rc}} = -\frac{kD}{r_{\mathrm{r}}^{\prime}} \cdot \frac{\omega_{\mathrm{s}}}{\omega_{\mathrm{b}}} i_{\mathrm{qs}}^{\mathrm{rc}}$$

$$\lambda_{\mathrm{dr}}^{\prime \mathrm{rc}} = 0$$

<div align="right">(628)</div>

It yields the slip angular velocity equation:

$$i_{\mathrm{dr}}^{\prime \mathrm{rc}} \cdot \frac{k^2 D}{x_{\mathrm{s}}} - \frac{x_{\mathrm{m}} kD}{x_{\mathrm{s}} r_{\mathrm{r}}^{\prime}} \cdot \frac{\omega_{\mathrm{s}}}{\omega_{\mathrm{b}}} i_{\mathrm{qs}}^{\mathrm{rc}} = 0$$

<div align="right">(629)</div>

which for a given rotor flux has always real solution. Thus the slip speed and the electromagnetic torque are:

$$\omega_{\mathrm{s}} = \frac{kr_{\mathrm{r}}^{\prime} \omega_{\mathrm{b}} i_{\mathrm{dr}}^{\prime \mathrm{rc}}}{x_{\mathrm{m}} i_{\mathrm{qs}}^{\mathrm{rc}}}$$

$$T_{\mathrm{e}} = \frac{P}{2} \cdot \frac{kx_{\mathrm{m}}}{\omega_{\mathrm{b}}} i_{\mathrm{qs}}^{\mathrm{rc}} i_{\mathrm{dr}}^{\mathrm{rc}}$$

<div align="right">(630-631)</div>

A comparison with the rotor current orientation control detailed for the three-phase induction machine permits to highlight the same conclusions. However, there has to be pointed out the necessary corrections:

- The transformation between synchronous reference frame and the stationary reference frame is made without changing the number of variables;
- The influence of the unsymmetrical stator windings configuration can be evidenced by using the turns ratio *k*, and it determines different effect over the rotor speed and torque response of the motor: higher electromagnetic torque and lower rotor speed for $k > 1$, respectively lower torque and higher rotor speed for $k < 1$.

4. MATHEMATICAL DISCRETE MODELS FOR THE THREE-PHASE INDUCTION MACHINE

4.1. Introduction

Advances in very-large-scale integration (VLSI) technology made possible the real-time modelling for many industrial applications. Real-time simulation is used increasingly in the automatic control field. One important application is the advanced AC motor control, i.e. vector control, where the immeasurable quantities, like the cage rotor parameters (flux, current), can be estimated by a simulator operating in parallel with the real motor. The simulation became an important alternative against the measurement, as the latter is complex, noise sensitive, and expensive. It has to be also mentioned that the approximate simulation always introduces some error between the true dynamic behaviour and the modelled behaviour (Vainio et al - 1992).

The continuous and discrete models of the induction machine are equivalent if their time response is similar for typical inputs (step or sinusoidal input). However, there are several discrete mathematical models equivalent to the continuous model. Iron loss and saturation can be omitted if the stator and rotor fluxes are limited to stay below the wide saturation border of the iron core, which is the case in most practical applications. For this equivalence, the approximation mode of the response for two consecutive sampling rates is determinant. Usually, when a time continuous system is discretized, the designer has to chose the properties of the system that will be maintained: the zero and pole number, the response characteristics for pulse, step and linear input, the DC amplify, the frequency response.

In Table 4.I there are presented several possible discretization methods for the time continuous systems.

TABLE 4.I

No	Discretization method	Continuous–discrete equivalence equation
1	Forward-difference method (Euler)	$s = \dfrac{1 - z^{-1}}{T \cdot z^{-1}}$
2	Backward-difference method	$s = \dfrac{1 - z^{-1}}{T}$
3	Bilinear transformation method (Tustin)	$s = \dfrac{2}{T} \cdot \dfrac{1 - z^{-1}}{1 + z^{-1}}$
4	Frequency prewarping method	$s = \dfrac{2}{T} \cdot \dfrac{1 - z^{-1}}{1 + z^{-1}}, \qquad \omega_A = \dfrac{2}{T} \cdot \tan \dfrac{\omega_D \cdot T}{2}$
5	Pulse invariance method	$G_D(z) = T \cdot Z\left\{ L^{-1}\left[G(s) \right] \right\}$
6	Step invariance method	$G_D(z) = \left(1 - z^{-1}\right) \cdot Z\left\{ L^{-1}\left[\dfrac{G(s)}{s} \right] \right\}$
7	Matched zero-pole method	Zero/pole from $s=-a$ is placed in discrete to $z=e^{-aT}$. Zero/pole from $\pm\infty$ is placed in discrete to $z=-1$

All of the above discretization methods are compared for the induction machine seen as a discrete system. The assumptions valid for the continuous model are also valid for the discrete systems. The results of the discretization are analysed from the point of view of computation complexity and the response stability for a step input signal. The analysis of the different models is essential for the design of electrical drive system with numerical command. A special attention is given to torque estimation from the data acquisition process (voltage or current quantities).

The synchronous reference frame is used for determining the discrete model of the induction machine. The following parameters can be defined:

$$\omega_k = \omega_s \quad \omega_s - \omega = \omega_r$$

For an easy to follow analysis, the index k will be omitted in the machine equations. The general form equations used to determine the discrete mathematical induction machine in *arbitrary synchronuos reference frames and per unit system* are:

$$\underline{U}_s(t) = r_s \underline{i}_s(t) + \frac{1}{\omega_n} \frac{d\underline{\psi}_s(t)}{dt} + j\omega_s \underline{\psi}_s(t) \tag{632}$$

$$0 = r'_r \underline{i}'_r(t) + \frac{1}{\omega_n} \frac{d\underline{\psi}'_r(t)}{dt} + j\omega_r \underline{\psi}'_r(t) \tag{633}$$

$$\underline{\psi}_s(t) = l_s \cdot \underline{i}_s(t) + \gamma \cdot l_M \cdot \underline{i}'_r(t) \tag{634}$$

$$\underline{\psi}_r(t) = \gamma^2 \cdot l'_r \cdot \underline{i}'_r(t) + \gamma \cdot l_M \cdot \underline{i}_s(t) \tag{635}$$

$$\underline{i}'_r = \frac{\underline{i}_r}{\gamma}; \quad \underline{\psi}'_r = \gamma \cdot \underline{\psi}_r; \quad r'_r = \gamma^2 \cdot r_r \tag{636}$$

$$\frac{d\omega}{dt} = \frac{1}{T_M} \cdot (t_e - t_l) \tag{637}$$

Depending on the synchronuous reference frame the following relations are valid:

a) Stator reference frame:

$$\gamma = \frac{l_s}{l_M}$$
$$t_e = \frac{3}{2} P \operatorname{Im}\{\underline{\psi}_s^* \cdot \underline{i}_s\} \tag{638}$$

b) Rotor reference frame:

$$\gamma = \frac{l_M}{l'_r}$$
$$t_e = \frac{3}{2} P \left(\frac{l_M}{l_r}\right) \operatorname{Im}\{\underline{\psi}_r^* \cdot \underline{i}_s\} \tag{639}$$

c) Airgap reference frame:

$$\gamma = 1$$
$$t_e = \frac{3}{2} P \operatorname{Im}\{\underline{\psi}_m^* \cdot \underline{i}_s\} \tag{640}$$
$$\underline{\psi}_m = l_M(\underline{i}_s + \underline{i}'_r)$$

If the determinant of the flux system equations is noted with δ and with d its inverse, the voltage equations are re-written as follows:

$$\underline{U}_s(t) = r_s \cdot d \cdot (\gamma \cdot l'_r \cdot \underline{\psi}_s - l_M \cdot \underline{\psi}'_r) + \frac{1}{\omega_n} \frac{d\underline{\psi}_s(t)}{dt} + j \cdot \omega_s \cdot \underline{\psi}_s(t) \tag{641}$$

$$0 = r'_r \cdot d \cdot (l_s \cdot \underline{\psi}'_r - \gamma \cdot l_M \cdot \underline{\psi}_s) + \frac{1}{\omega_n} \frac{d\underline{\psi}'_r(t)}{dt} + j \cdot \omega_r \cdot \underline{\psi}'_r(t) \tag{642}$$

For zero initial conditions, if the Laplace transformation is applied, it results the matrix system:

$$\frac{s}{\omega_n} \cdot \begin{bmatrix} \underline{\psi}_s \\ \underline{\psi}'_r \end{bmatrix} = \begin{bmatrix} -\gamma \cdot r_s l'_r d - j\omega_s & r_s l_M d \\ \gamma \cdot r'_r l_M d & -r'_r l_s d - j\omega_r \end{bmatrix} \cdot \begin{bmatrix} \underline{\psi}_s \\ \underline{\psi}'_r \end{bmatrix} + \begin{bmatrix} \underline{U}_s \\ 0 \end{bmatrix} \tag{643}$$

or in synthetisized form:

$$s \cdot Y(s) = A \cdot Y(s) + B \cdot U(s) \tag{644}$$

One can note that by comparing with the state variable system described by the equations:

$$\begin{cases} s \cdot X(s) = A \cdot X(s) + B \cdot U(s) \\ Y(s) = C \cdot X(s) + D \cdot U(s) \end{cases} \tag{645}$$

the state variable vector is identical to the output vector (fluxes vector in this case). The input vector is the voltage vector. Through identification we obtain:

$$A = \omega_n \cdot \begin{bmatrix} -\gamma \cdot r_s l'_r d - j\omega_s & r_s l_M d \\ \gamma \cdot r'_r l_M d & -r'_r l_s d - j\omega_r \end{bmatrix} \tag{646}$$

$$B = \omega_n$$

4.2. Bilinear transformation method (Tustin)

The relation (3) from the Table I is known as bilinear transformation or Tustin and it makes the connection between continuous to discrete domain. This method gives better results than the trapezoidal approximation, or other discretization method illustrated in Table 4.II, by considering first four order of integration operators. The describing system relation becomes:

$$\frac{2}{T} \cdot \frac{z-1}{z+1} \cdot Y(z) = A \cdot Y(z) + B \cdot U(z) \tag{647}$$

where T is the signal sampling period for continuous domain, and f is the sampling frequency.

TABLE 4.II

Method	$\dfrac{1}{s}$	$\dfrac{1}{s^2}$	$\dfrac{1}{s^3}$	$\dfrac{1}{s^4}$
Tustin	$\dfrac{T}{2} \cdot \dfrac{z+1}{z-1}$	$\dfrac{T^2}{4} \cdot \dfrac{(z+1)^2}{(z-1)^2}$	$\dfrac{T^3}{8} \cdot \dfrac{(z+1)^3}{(z-1)^3}$	$\dfrac{T^4}{16} \cdot \dfrac{(z+1)^4}{(z-1)^4}$
Boxer-Thaler	$\dfrac{T}{2} \cdot \dfrac{z+1}{z-1}$	$\dfrac{T^2}{12} \cdot \dfrac{z^2+10z+1}{(z-1)^2}$	$\dfrac{T^3}{2} \cdot \dfrac{z^2+z}{(z-1)^3}$	$\dfrac{T^4}{6} \cdot \dfrac{z^3+4z^2+z}{(z-1)^4}$
Madwed	$\dfrac{T}{2} \cdot \dfrac{z+1}{z-1}$	$\dfrac{T^2}{6} \cdot \dfrac{z^2+4z+1}{(z-1)^2}$	$\dfrac{T^3}{24} \cdot \dfrac{z^3+11z^2+11z+1}{(z-1)^3}$	$\dfrac{T^4}{120} \cdot \dfrac{z^4+26z^3+66z^2+1}{(z-1)^4}$

$$\frac{2}{T} \cdot (z-1) \cdot Y(z) = (z+1) \cdot A \cdot Y(z) + (z+1) \cdot B \cdot U(z)$$

$$\frac{2}{T} \cdot z Y(z) - \frac{2}{T} \cdot Y(z) = z \cdot A \cdot Y(z) + A \cdot Y(z) + (z+1) \cdot B \cdot U(z)$$

$$\left(\frac{2}{T} \cdot I - A\right) \cdot z \cdot Y(z) = \left(\frac{2}{T} \cdot I + A\right) \cdot Y(z) + (z+1) \cdot B \cdot U(z)$$

$$z \cdot Y(z) = \left(\frac{2}{T} \cdot I - A\right)^{-1} \cdot \left(\frac{2}{T} \cdot I + A\right) \cdot Y(z) + \left(\frac{2}{T} \cdot I - A\right)^{-1} \cdot B \cdot (z+1) \cdot U(z)$$

$$Y(z) = \left(\frac{2}{T} \cdot I - A\right)^{-1} \cdot \left(\frac{2}{T} \cdot I + A\right) \cdot z^{-1} \cdot Y(z) + \left(\frac{2}{T} \cdot I - A\right)^{-1} \cdot B \cdot (1 + z^{-1}) \cdot U(z)$$

$$Y(z) = \frac{1}{\Delta} \cdot C \cdot z^{-1} \cdot Y(z) + \frac{1}{\Delta} \cdot D \cdot (1 + z^{-1}) \cdot U(z)$$

$$(648)$$

where Δ is the matrix determinant: $\left[\dfrac{2}{T} \cdot I - A\right]$:

$$\Delta = \Delta_R + j \cdot \Delta_I = (2f + \omega_n \cdot \gamma \cdot r_s l'_r d) \cdot (2f + \omega_n \cdot r'_r l_s d) - \omega_n^2 \cdot (\gamma \cdot r_s r'_r l_M^2 d^2 + \omega_r \omega_s) +$$
$$+ j \cdot \omega_n \cdot [\omega_r \cdot (2f + \omega_n \cdot \gamma \cdot r_s l'_r d) + \omega_s \cdot (2f + \omega \cdot r'_r l_s d)] \tag{649}$$

$$c_{11} = (2f - \omega_n \cdot \gamma \cdot r_s l'_r d) \cdot (2f + \omega_n \cdot r'_r l_s d) + \omega_n^2 \cdot (\gamma \cdot r_s r'_r l_M^2 d^2 + \omega_r \omega_s) +$$
$$+ j \cdot \omega_n \cdot [\omega_r \cdot (2f - \omega_n \gamma \cdot r_s l'_r d) - \omega_s \cdot (2f + \omega_n r'_r l_s d)]$$
$$c_{12} = 4f \omega_n \gamma \cdot r_s l_M d$$
$$c_{21} = 4f \omega_n r'_r l_M d \tag{650}$$
$$c_{22} = (2f + \omega_n \cdot \gamma \cdot r_s l_r d) \cdot (2f - \omega_n \cdot r'_r l_s d) + \omega_n^2 \cdot (\gamma \cdot r_s r_r l_M^2 d^2 + \omega_r \omega_s) +$$
$$+ j \cdot \omega_n \cdot [-\omega_r \cdot (2f + \omega_n \gamma \cdot r_s l_r d) + \omega_s \cdot (2f - \omega_n r'_r l_s d)]$$
$$d_{11} = \omega_n \cdot (2f + \omega_n r'_r l_s d) + j \cdot \omega_r \omega_n^2$$
$$d_{21} = r'_r l_M d \omega_n^2$$

Observation: d_{12}, and d_{22} are not calculated as the induction machine is considered to have cage rotor.

$$\begin{cases} \underline{\psi}_s(n) = \dfrac{1}{\Delta} \cdot [c_{11} \cdot \underline{\psi}_s(n-1) + c_{12} \cdot \underline{\psi}'_r(n-1)] + \dfrac{1}{\Delta} \cdot d_{11} \cdot [\underline{U}_s(n-1) + \underline{U}_s(n)] \\ \underline{\psi}'_r(n) = \dfrac{1}{\Delta} \cdot [c_{21} \cdot \underline{\psi}_s(n-1) + c_{22} \cdot \underline{\psi}'_r(n-1)] + \dfrac{1}{\Delta} \cdot d_{21} \cdot [\underline{U}_s(n-1) + \underline{U}_s(n)] \end{cases} \tag{651}$$

where n and $n-1$ are two consecutive sampling periods in discrete domain.

The induction machine discrete model is presented in Fig. 4.1. For this model the stator voltage represents the input vector and the fluxes are the expressed as output vector.
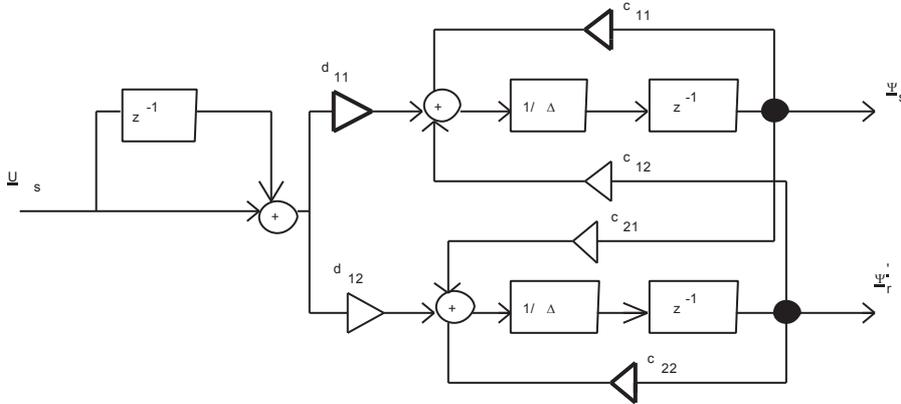
Fig. 4.1. The induction machine discrete model

The blocks with thicker lines denote complex multipliers. The following matrix relation is obtained from the separation of the real and the imaginary parts of the flux equations.

$$\begin{bmatrix} \psi_{sR}(n) \\ \psi_{sI}(n) \\ \psi'_{rR}(n) \\ \psi'_{rI}(n) \end{bmatrix} = \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot E \cdot \begin{bmatrix} \psi_{sR}(n-1) \\ \psi_{sI}(n-1) \\ \psi'_{rR}(n-1) \\ \psi'_{rI}(n-1) \end{bmatrix} + \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot F \cdot \begin{bmatrix} U_{sR}(n-1) + U_{sR}(n) \\ U_{sI}(n-1) + U_{sI}(n) \\ 0 \\ 0 \end{bmatrix} \tag{652}$$

where:

$$e_{11} = e_{22} = c_{11R}\Delta_R + c_{11I}\Delta_I$$
$$e_{12} = -e_{21} = c_{11R}\Delta_I - c_{11I}\Delta_R$$
$$e_{13} = e_{24} = c_{12R}\Delta_R$$
$$e_{14} = -e_{23} = c_{12R}\Delta_I$$
$$e_{31} = e_{42} = c_{21R}\Delta_R$$
$$e_{32} = -e_{41} = c_{21R}\Delta_I$$
$$e_{33} = e_{44} = c_{22R}\Delta_R + c_{22I}\Delta_I$$
$$e_{34} = -e_{43} = c_{22R}\Delta_I - c_{22I}\Delta_R \qquad (653)$$
$$f_{11} = f_{22} = d_{11R}\Delta_R + d_{11I}\Delta_R$$
$$f_{12} = -f_{21} = d_{11R}\Delta_I - d_{11I}\Delta_R$$
$$f_{31} = f_{42} = d_{21R}\Delta_R$$
$$f_{32} = -f_{41} = d_{21R}\Delta_I$$

The complete discrete model of the induction machine, based on the bilinear method is given in Fig. 4.2. The electromagnetic torque expressed in relative units is considering different reference frames for the induction machine:

a) Stator reference frame

$$t_e = \frac{3}{2}P\,\mathrm{Im}\{\underline{\psi}_s^* \cdot \underline{i}_s\} = \frac{3}{2}P\,\mathrm{Im}\{(\psi_{sR} - j\psi_{sI})\cdot(i_{sR} + j\cdot i_{sI})\} = \frac{3}{2}P(\psi_{sR}\cdot i_{sI} - \psi_{sI}\cdot i_{sR}) =$$
$$= \frac{3}{2}P(\psi_{sR}\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sI} - l_M d\cdot\psi'_{rI}) - \psi_{sI}\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sR} - l_M d\cdot\psi'_{rR})) = \frac{3}{2}P\cdot l_M d\cdot(\psi_{sI}\cdot\psi'_{rR} - \psi_{sR}\cdot\psi'_{rI})$$
$$(654)$$

b) Rotor reference frame:

$$t_e = \frac{3}{2}P\cdot\gamma\cdot\mathrm{Im}\{\underline{\psi}_r^* \cdot \underline{i}_s\} = \frac{3}{2}P\cdot\gamma\cdot\mathrm{Im}\{(\psi'_{rR} - j\cdot\psi'_{rI})\cdot(i_{sR} + j\cdot i_{sI})\} = \frac{3}{2}P\cdot\gamma\cdot(\psi'_{rR}\cdot i_{sI} - \psi'_{rI}\cdot i_{sR}) =$$
$$= \frac{3}{2}P\cdot\gamma\cdot(\psi'_{rR}\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sI} - l_M d\cdot\psi'_{rI}) - \psi'_{rI}\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sR} - l_M d\cdot\psi'_{rR})) = \frac{3}{2}P\cdot\gamma^2\cdot l'_r\,d\cdot(\psi_{sI}\cdot\psi'_{rR} - \psi_{sR}\cdot\psi'_{rI}) =$$
$$= \frac{3}{2}P\cdot l_M d\cdot(\psi_{sI}\cdot\psi'_{rR} - \psi_{sR}\cdot\psi'_{rI})$$
$$(655)$$

c) Airgap reference frame:

$$t_e = \frac{3}{2}P\,\mathrm{Im}\{\underline{\psi}_m^* \cdot \underline{i}_s\} = \frac{3}{2}P\,\mathrm{Im}\{(\psi_{mR} - j\cdot\psi_{mI})\cdot(i_{sR} + j\cdot i_{sI})\} = \frac{3}{2}P(\psi_{mR}\cdot i_{sI} - \psi_{mI}\cdot i_{sR}) =$$
$$= \frac{3}{2}P(\psi_{mR}\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sI} - l_M d\cdot\psi'_{rI}) - \psi_{mI}\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sR} - l_M d\cdot\psi'_{rR})) =$$
$$= \frac{3}{2}P\left(\begin{array}{c} l_M d\cdot(\psi_{sR}\cdot\gamma\cdot(l'_r - \gamma\cdot l_M) + \psi'_{rR}(\gamma\cdot l_s - l_M))\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sI} - l_M d\cdot\psi'_{rI}) - \\ -l_M d\cdot(\psi_{sI}\cdot\gamma\cdot(l'_r - \gamma\cdot l_M) + \psi'_{rI}(\gamma\cdot l_s - l_M))\cdot(\gamma\cdot l'_r\,d\cdot\psi_{sR} - l_M d\cdot\psi'_{rR}) \end{array}\right) =$$
$$= \frac{3}{2}P\cdot l_M d\cdot\left[\begin{array}{c} (\psi_{sR}\cdot(l'_r - l_M) + \psi'_{rR}(l_s - l_M))\cdot(l'_r\,d\cdot\psi_{sI} - l_M d\cdot\psi'_{rI}) - \\ -(\psi_{sI}\cdot(l'_r - l_M) + \psi'_{rI}(l_s - l_M))\cdot(l'_r\,d\cdot\psi_{sR} - l_M d\cdot\psi'_{rR}) \end{array}\right] = \frac{3}{2}P\cdot l_M d\cdot(\psi_{sI}\cdot\psi'_{rR} - \psi_{sR}\cdot\psi'_{rI})$$
$$(656)$$

From the torque equations, written in stator, rotor and air-gap reference frames, one can note that the same general relation can be implemented, though the equivalent rotor flux stands for different significance, i.e. only in air-gap reference frame it has the real physical rotor flux.

As it can be observed by studying the block diagram of the discrete model of the induction machine, for a complete implementation there are necessary 31 multiplier blocks, 23 summing blocks and 4 delay blocks. However, this model can be simplified further. The total number of multiplier blocks can vary from one implementation to another, due to the place of this operation in the block diagram. Also the frequency response is variable according to the implementation version (Vainio et al - 1992).
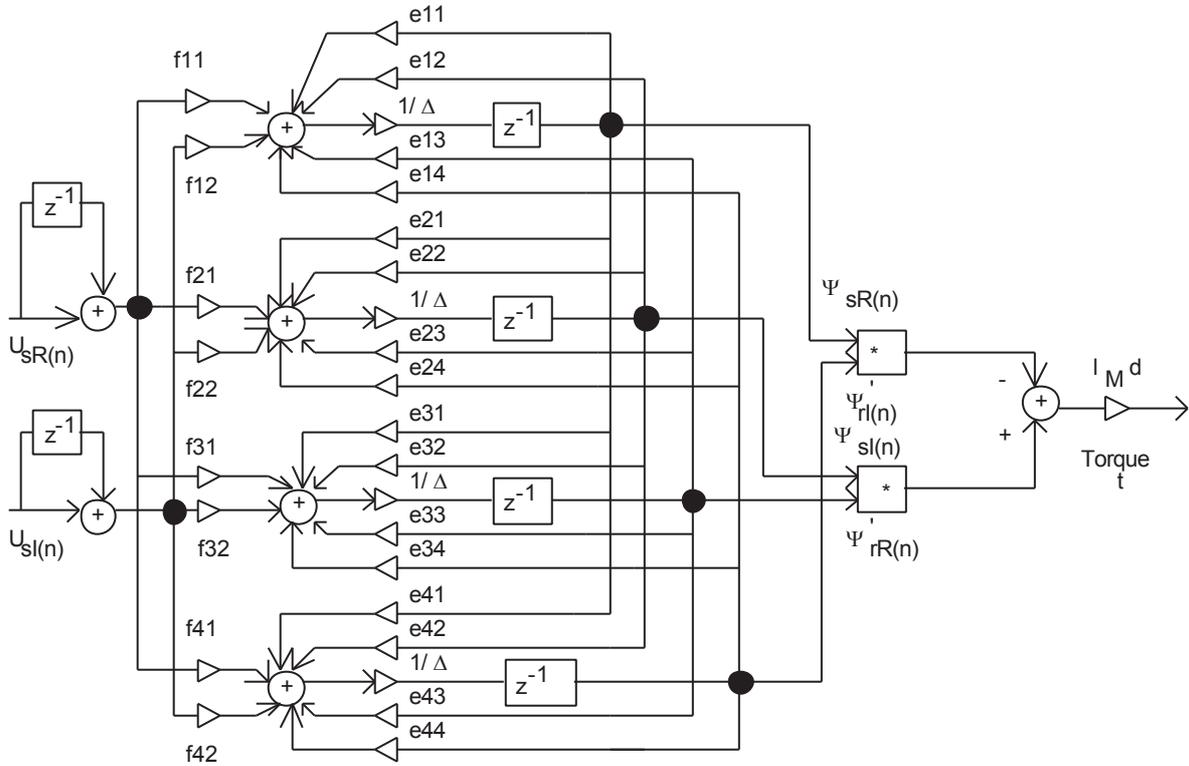

Fig. 4.2. Complete discrete induction machine model based on
bilinear transformation method (Tustin)

4.3. Forward-differences method (Euler)

In this method the continuous-time derivative is approximated by the relation (1) from Table 4.I or by a scaled difference of two successive samples:

$$\frac{dX}{dt} = \frac{X(n+1) - X(n)}{T} \tag{657}$$

where T is the sampling period. The discrete-time model is in this case defined by the following equations:

$$\underline{U}_s(n) = r_s \cdot \left(\gamma \cdot l'_r \, d \cdot \underline{\psi}_s(n) - l_M d \cdot \underline{\psi}'_r(n) \right) + \frac{1}{\omega_n} \cdot \frac{\underline{\psi}_s(n+1) - \underline{\psi}_s(n+1)}{T} + j \cdot \omega_s \cdot \underline{\psi}_s(n) \tag{658}$$

$$0 = r'_r \cdot \left(l_s d \cdot \underline{\psi}'_r(n) - \gamma \cdot l_M d \cdot \underline{\psi}_s(n) \right) + \frac{1}{\omega_n} \cdot \frac{\underline{\psi}'_r(n+1) - \underline{\psi}_r(n+1)}{T} + j \cdot \omega_r \cdot \underline{\psi}_r(n) \tag{659}$$

In matrix form it results:

$$\begin{bmatrix} \underline{\psi}_s(n+1) \\ \underline{\psi}'_r(n+1) \end{bmatrix} = \omega_n T \cdot \begin{bmatrix} -\gamma \cdot r_s l'_r d \cdot \dfrac{1}{\omega_n T} - j \cdot \omega_s & r_s l_M d \\ \gamma \cdot r'_r l_M d & -r'_r l_s d \cdot \dfrac{1}{\omega_n T} - j \cdot \omega_r \end{bmatrix} \cdot \begin{bmatrix} \underline{\psi}_s(n) \\ \underline{\psi}'_r(n) \end{bmatrix} + \omega_n T \cdot \begin{bmatrix} \underline{U}_s(n) \\ 0 \end{bmatrix}$$

(660)

After separating the real and the imaginary components, it will result:

$$\psi_{sR}(n+1) + j \cdot \psi_{sI}(n+1) = \omega_n T \cdot (-\gamma \cdot r_s l'_r d + \frac{1}{\omega_n T} - j \cdot \omega_s) \cdot [\psi_{sR}(n) + j \cdot \psi_{sI}(n)] +$$

(661)

$$+\omega_n T \cdot \gamma \cdot r_s l_M d \cdot [\psi'_{rR}(n) + j \cdot \psi'_{rI}(n)] + \omega_n T \cdot [U_{sR}(n) + j \cdot U_{sI}(n)]$$

$$\psi'_{rR}(n+1) + j \cdot \psi'_{rI}(n+1) = \omega_n T \cdot \gamma \cdot r'_r l_M d \cdot [\psi_{sR}(n) + j \cdot \psi_{sI}(n)] +$$

$$+\omega_n T \cdot (-r'_r l_s d + \frac{1}{\omega_n T} - j \cdot \omega_r) \cdot [\psi'_{rR}(n) + j \cdot \psi'_{rI}(n)]$$

(662)

$$\begin{bmatrix} \psi_{sR}(n+1) \\ \psi_{sI}(n+1) \\ \psi'_{rR}(n+1) \\ \psi'_{rI}(n+1) \end{bmatrix} = \begin{bmatrix} 1-\omega_n T \cdot \gamma \cdot r_s l'_r d & \omega_n T \omega_s & \omega_n T \cdot r_s l_M d & 0 \\ -\omega_n T \cdot \omega_s & 1-\omega_n T \cdot \gamma \cdot r_s l'_r d & 0 & \omega_n T \cdot r_s l_M d \\ \omega_n T \cdot \gamma \cdot r'_r l_M d & 0 & 1-\omega_n T \cdot r'_r l_s d & \omega_n T \cdot \omega_r \\ 0 & \omega_n T \cdot \gamma \cdot r'_r l_M d & -\omega_n T \cdot \omega & 1-\omega_n T \cdot r'_r l_s d \end{bmatrix} \begin{bmatrix} \psi_{sR}(n) \\ \psi_{sI}(n) \\ \psi_{rR}(n) \\ \psi_{rI}(n) \end{bmatrix} + \omega_n T \cdot \begin{bmatrix} U_{sR}(n) \\ U_{sI}(n) \\ 0 \\ 0 \end{bmatrix}$$

(663)

or:

$$\psi(n+1) = A \cdot \psi(n) + \omega_n T \cdot U(n)$$

(664)

For the current vector is valid the following relation:

$$i(n) = B \cdot \psi(n)$$

(665)

or the matrix form equation:

$$\begin{bmatrix} i_{sR}(n) \\ i_{sI}(n) \\ i'_{rR}(n) \\ i'_{rI}(n) \end{bmatrix} = \begin{bmatrix} \gamma \cdot l'_r d & 0 & -l_M d & 0 \\ 0 & \gamma \cdot l'_r d & 0 & -l_M d \\ -\gamma \cdot l_M d & 0 & l_s d & 0 \\ 0 & -\gamma \cdot l_M d & 0 & l_s d \end{bmatrix} \cdot \begin{bmatrix} \psi_{sR}(n) \\ \psi_{sI}(n) \\ \psi'_{rR}(n) \\ \psi'_{rI}(n) \end{bmatrix}$$

(666)

For this case the complete discrete model of the induction machine is described in Fig. 4.3.
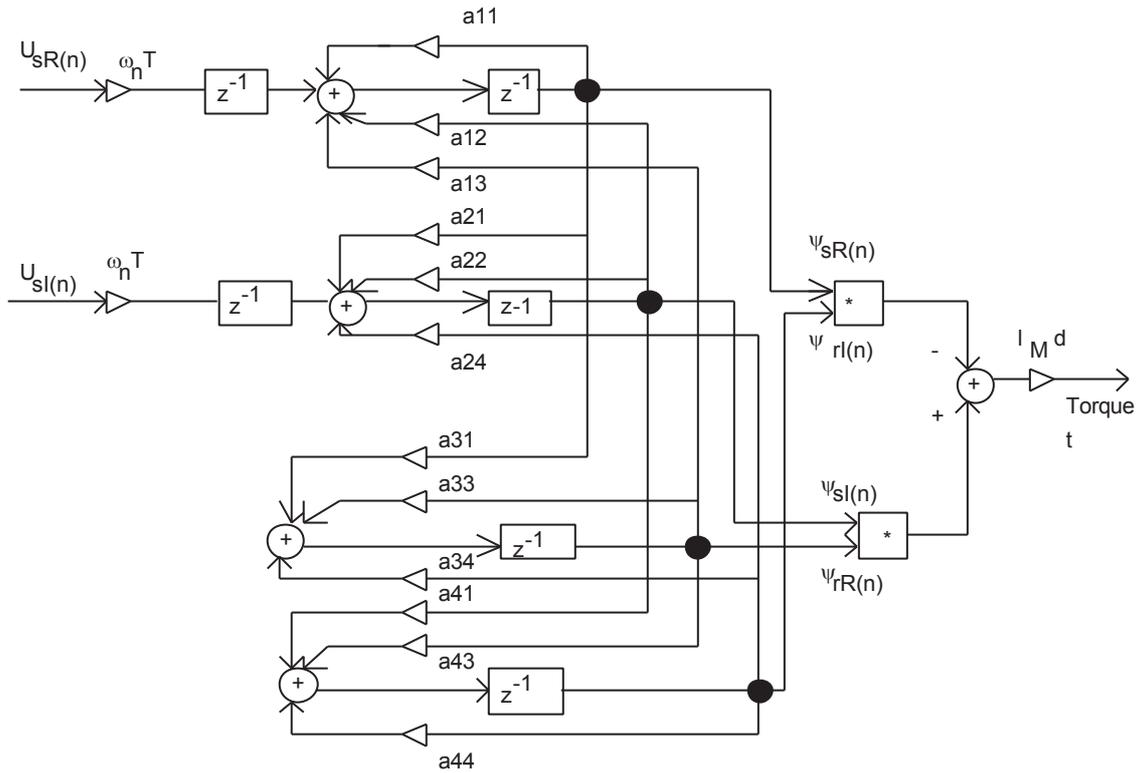
Fig. 4.3. Complete discrete induction machine model based on
forward-differences method (Euler)

4.4. Backward-differences method

This simple method produces a stabile discrete system for a stabile time continuous system. Even some unstable continuous systems can be transformed to stabile version through this discretization method. However, a higher sampling frequency has to be adopted in order to avoid the frequency response distortions of the system. With relation (2) from Table 4.I the following equations can be implemented:

$$\frac{1-z^{-1}}{T} \cdot \boldsymbol{Y} = \boldsymbol{A} \cdot \boldsymbol{Y} + \boldsymbol{B} \cdot \boldsymbol{U} \tag{667}$$

$$(\boldsymbol{I} - \boldsymbol{A} \cdot T) \cdot \boldsymbol{Y} = z^{-1} \cdot \boldsymbol{Y} + \boldsymbol{B} \cdot T \cdot \boldsymbol{U}$$

$$\boldsymbol{I} - \boldsymbol{A} \cdot T = \begin{bmatrix} 1 + \omega_n T\gamma \cdot r_s l'_r d + j\omega_s \omega_n T & -\omega_n T r_s l_M d \\ -\omega_n T\gamma \cdot r'_r l_M d & 1 + \omega_n T r'_r l_s d + j\omega_r \omega_n T \end{bmatrix} \tag{668}$$

$$\Delta_{I-AT} = \Delta_R + j\Delta_I = (1 + \omega_n T\gamma \cdot r_s l'_r d + j\omega_s \omega_n T) \cdot (1 + \omega_n T r'_r l_s d + j\omega_r \omega_n T) - \omega_n^2 T^2 \gamma \cdot r_s r'_r l_M^2 d^2 =$$

$$= (1 + \omega_n T\gamma \cdot r_s l'_r d) \cdot (1 + \omega_n T r'_r l_s d) - \omega_n^2 T^2 \cdot (\omega_s \omega_r - \gamma \cdot r_s r'_r l_M^2 d^2) +$$

$$+ j\omega_n T \cdot \left[\omega_r \cdot (1 + \omega_n T\gamma \cdot r_s l'_r d) + \omega_s \cdot (1 + \omega_n T r'_r l_s d) \right] \tag{669}$$

$$(\boldsymbol{I} - \boldsymbol{A} \cdot T)^{-1} = \frac{1}{\Delta_{I-AT}} \cdot \begin{bmatrix} 1 + \omega_n T r'_r l_s d + j\omega_r \omega_n T & \omega_n T r_s l_M d \\ \omega_n T\gamma \cdot r'_r l_M d & 1 + \omega_n T\gamma \cdot r_s l'_r d + j\omega_s \omega_n T \end{bmatrix} \tag{670}$$

By substituting vectors $\boldsymbol{Y}$ and $\boldsymbol{U}$ with the flux vector, respectively stator voltage, it follows that:

$$\begin{bmatrix} \underline{\psi}_s(n) \\ \underline{\psi}'_r(n) \end{bmatrix} = (\boldsymbol{I} - \boldsymbol{A} \cdot T)^{-1} \cdot \begin{bmatrix} \underline{\psi}_s(n-1) \\ \underline{\psi}'_r(n-1) \end{bmatrix} + (\boldsymbol{I} - \boldsymbol{A} \cdot T)^{-1} \cdot \boldsymbol{B} \cdot T \cdot \begin{bmatrix} \underline{U}_s(n) \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\Delta_R{}^2 + \Delta_I{}^2} \cdot \boldsymbol{G} \cdot \begin{bmatrix} \underline{\psi}_s(n-1) \\ \underline{\psi}'_r(n-1) \end{bmatrix} + \frac{1}{\Delta_R{}^2 + \Delta_I{}^2} \cdot \boldsymbol{H} \cdot \begin{bmatrix} \underline{U}_s(n) \\ 0 \end{bmatrix}$$

(671)

When the real and the imaginary parts are separated, it results that:

$$\begin{bmatrix} \psi_{sR}(n) \\ \psi_{sI}(n) \\ \psi'_{rR}(n) \\ \psi'_{rI}(n) \end{bmatrix} = \frac{1}{\Delta_R{}^2 + \Delta_I{}^2} \cdot \boldsymbol{L} \cdot \begin{bmatrix} \psi_{sR}(n-1) \\ \psi_{sI}(n-1) \\ \psi'_{rR}(n-1) \\ \psi'_{rI}(n-1) \end{bmatrix} + \frac{1}{\Delta_R{}^2 + \Delta_I{}^2} \cdot \boldsymbol{M} \cdot \begin{bmatrix} U_{sR}(n) \\ U_{sI}(n) \\ 0 \\ 0 \end{bmatrix}$$

(672)

$$l_{11} = l_{22} = g_{11R}\Delta_R + g_{11I}\Delta_I$$
$$l_{12} = -l_{21} = g_{11R}\Delta_I - g_{11I}\Delta_R$$
$$l_{13} = l_{24} = g_{12}\Delta_R$$
$$l_{14} = -l_{23} = g_{12}\Delta_I$$
$$l_{31} = l_{42} = g_{21}\Delta_R$$
$$l_{32} = -l_{41} = g_{21}\Delta_I$$
$$l_{33} = l_{44} = g_{22R}\Delta_R + g_{22I}\Delta_I$$
$$l_{34} = -l_{43} = g_{22R}\Delta_I + g_{22I}\Delta_R$$
$$m_{11} = m_{22} = h_{11R}\Delta_R + h_{11I}\Delta_I$$
$$m_{12} = -m_{12} = h_{11R}\Delta_I - h_{11I}\Delta_R$$
$$m_{31} = m_{42} = h_{21R}\Delta_R$$
$$m_{32} = -m_{41} = h_{21R}\Delta_I$$

(673)

The complete discrete mathematical model of the machine, obtained through the backward-difference method is presented in Fig. 4.4. A similar analysis was made for the step and pulse invariance methods by Vainio et al (1992). The results permit some important conclusions to be drawn:

- The pulse invariance method and backward-difference method determine similar and very resembling discrete mathematical model for the machine;
- The step invariance method and the forward-difference method determine similar and very resembling mathematical discrete model for the machine.

Nevertheless, although the methods are very similar in results, the frequency characteristics of the model differ from one method to another.

The computational burden for the three described methods is summarised in Table 4.III. As the DSP-ASIC implementation is an important cost issue, the designer has to choose a compromise between the complexity and the accuracy of the model. When the comparison is made, one can note the advantage of the forward-difference method (Euler) for less computing time. When accuracy is the determining factor, the bilinear transformation method (Tustin) has to be chosen for the implementation of the mathematical discrete model of the machine. It is also possible to apply a hybrid approach where the stator equation is discretized using the forward-difference (Euler) method, whereas the rotor equation is converted using the bilinear transformation (Tustin) or vice-versa. As one might expect, the resulting response and computational complexity are between those of the complete methods. This could be exploited when implementing the discretized model with a programmable signal processor.

TABLE 4.III

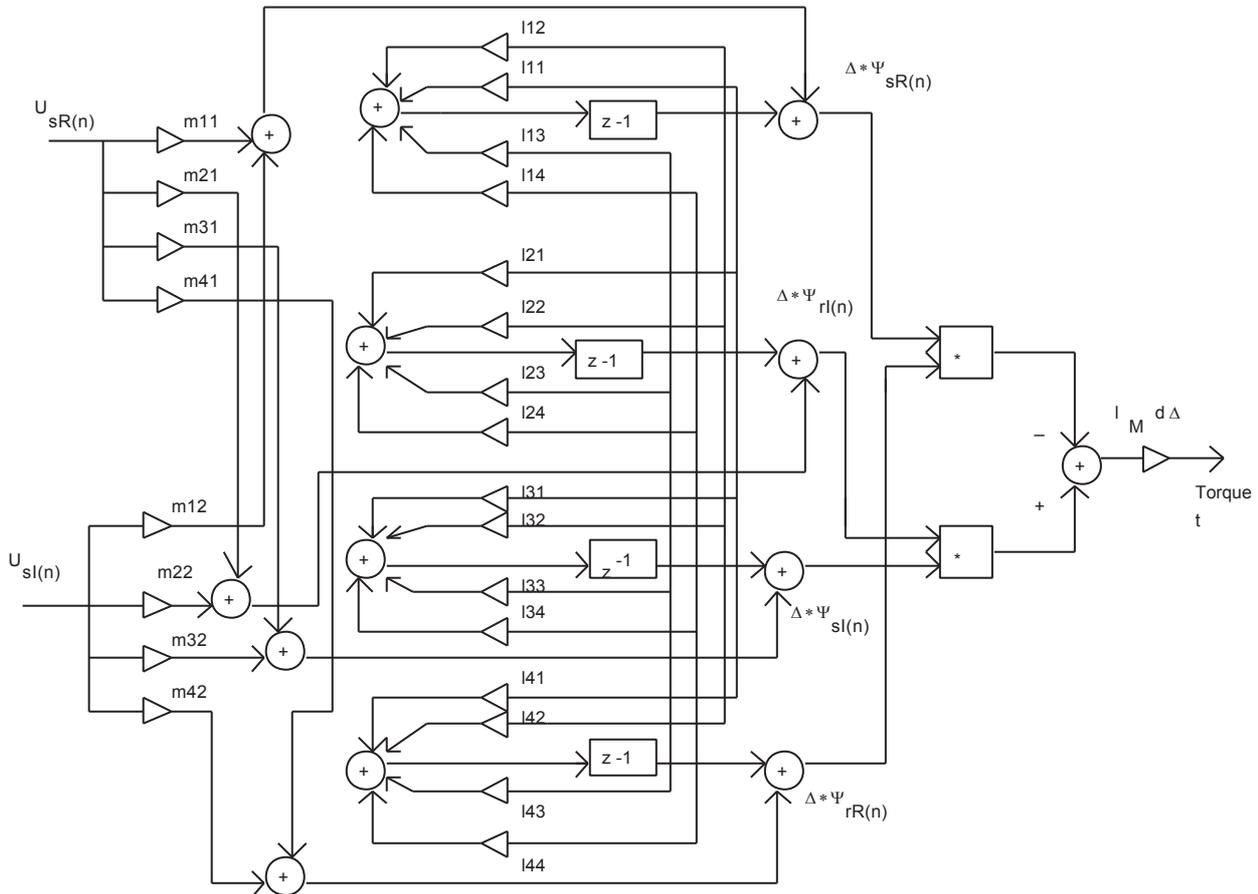| Method \ Real operation	Additions	Multiplications	Delays
Forward-difference method (Euler)	11	17	6
Backward-difference method	21	27	4
Bilinear transformation method (Tustin)	23	31	4

Fig. 4.4. Complete discrete induction machine model based on
backward-differences method

4.5. Z-domain transfer functions

The general form of a state-variable system using matrix notation in z-domain is:

$$x_s(n+1) = A \cdot x_s(n) + B \cdot u_s(n)$$

$$y_s(n) = C \cdot x_s(n) + D \cdot u_s(n) \tag{674}$$

$$x_s(n) = \left[x_1(n),\ ...,\ x_m(n)\right]^{\mathrm{T}}$$

where $x_s(n)$ is the vector for the state variables, $u_s(n)$ and $y_s(n)$ are the input, respectively the output vectors, and A is the state matrix. The impulse response sequence in terms of the state-variable description is given by the relation:

$$h(k) = \begin{cases} D, & \text{for } k = 0 \\ C \cdot A^{k-1}, & \text{for } k > 0 \end{cases} \tag{675}$$

and the transfer function matrix:

$$H_{\psi}(z) = \sum_{k=0}^{\infty} h(k) \cdot z^{-k} = D + C \cdot (zI - A)^{-1} \cdot B \tag{676}$$

For the induction machine model case, there are four real outputs (the stator and rotor fluxes) and two real inputs (stator voltage). Also, the state variables are the outputs directly. We can establish the notation:

$$x_s(n) = y_s(n) = \begin{bmatrix} \psi_{sR}(n) \\ \psi_{sI}(n) \\ \psi_{rR}(n) \\ \psi_{rI}(n) \end{bmatrix}$$

The state matrix A, the input vector u_s and the coefficients matrix B, C, and D have different element according to the transformation method used for implementing the discrete mathematical machine model.

I) Bilinear transformation method (Tustin):
The state matrix A is calculated as follows:

$$A = \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot \begin{bmatrix} c_{11R}\Delta_R + c_{11I}\Delta_I & c_{11R}\Delta_I & c_{12R}\Delta_I & c_{12R}\Delta_I \\ -c_{11R}\Delta_I & c_{11R}\Delta_R + c_{11I}\Delta_I & -c_{12R}\Delta_I & c_{12R}\Delta_I \\ c_{21R}\Delta_R & c_{21R}\Delta_I & c_{22R}\Delta_R + c_{22I}\Delta_I & c_{22R}\Delta_I - c_{22I}\Delta_R \\ -c_{21R}\Delta_I & c_{21R}\Delta_R & -c_{22R}\Delta_I + c_{22I}\Delta_R & c_{22R}\Delta_R + c_{22I}\Delta_I \end{bmatrix} \tag{677}$$

where:

$$\Delta = \Delta_R + j \cdot \Delta_I = (2f + \omega_n \cdot \gamma \cdot r_s l'_r d) \cdot (2f + \omega_n \cdot r'_r l_s d) - \omega_n^2 \cdot (\gamma \cdot r_s r'_r l_M^2 d^2 + \omega_r \omega_s) + \\ + j \cdot \omega_n \cdot [\omega_r \cdot (2f + \omega_n \cdot \gamma \cdot r_s l'_r d) + \omega_s \cdot (2f + \omega \cdot r'_r l_s d)] \tag{678}$$

$$c_{11} = (2f - \omega_n \cdot \gamma \cdot r_s l'_r d) \cdot (2f + \omega_n \cdot r'_r l_s d) + \omega_n^2 \cdot (\gamma \cdot r_s r'_r l_M^2 d^2 + \omega_r \omega_s) + \\ + j \cdot \omega_n \cdot [\omega_r \cdot (2f - \omega_n \gamma \cdot r_s l'_r d) - \omega_s \cdot (2f + \omega_n r'_r l_s d)]$$
$$c_{12} = 4f\omega_n \gamma \cdot r_s l_M d$$
$$c_{21} = 4f\omega_n r'_r l_M d \tag{679}$$
$$c_{22} = (2f + \omega_n \cdot \gamma \cdot r_s l'_r d) \cdot (2f - \omega_n \cdot r'_r l_s d) + \omega_n^2 \cdot (\gamma \cdot r_s r'_r l_M^2 d^2 + \omega_r \omega_s) + \\ + j \cdot \omega_n \cdot [-\omega_r \cdot (2f + \omega_n \gamma \cdot r_s l'_r d) + \omega_s \cdot (2f - \omega_n r'_r l_s d)]$$

The input vector is determined with the relation:

$$u_s = \begin{bmatrix} U_{sR}(n+1) - U_{sR}(n) \\ U_{sI}(n+1) - U_{sI}(n) \end{bmatrix} \tag{680}$$

The coefficients matrix are:

$$B = \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot \begin{bmatrix} d_{11R}\Delta_R + d_{11I}\Delta_I & d_{11R}\Delta_I - d_{11I}\Delta_R \\ -d_{11R}\Delta_I + d_{11I}\Delta_R & d_{11R}\Delta_R + d_{11I}\Delta_I \\ d_{21R}\Delta_I & d_{21R}\Delta_I \\ -d_{21R}\Delta_I & d_{21R}\Delta_I \end{bmatrix} \tag{681}$$

where:

$$d_{11} = \omega_n \cdot (2f + \omega_n r'_r l_s d) + j \cdot \omega_r \omega_n^2$$
$$d_{21} = r'_r l_M d \omega_n^2$$

By identification, we establish the other coefficients matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{682}$$

From the transfer function matrix relation, we obtain a 4 X 2 matrix giving the transfer function from both inputs to the four outputs:

$$H_\psi(z) = \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot \begin{bmatrix} z - \dfrac{c_{11R}\Delta_R + c_{11I}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{c_{11R}\Delta_I}{\Delta_R^2 + \Delta_I^2}{}_I & -\dfrac{c_{12R}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{c_{12R}\Delta_I}{\Delta_R^2 + \Delta_I^2} \\ \dfrac{c_{11R}\Delta_I}{\Delta_R^2 + \Delta_I^2} & z - \dfrac{c_{11R}\Delta_R + c_{11I}\Delta_I}{\Delta_R^2 + \Delta_I^2} & \dfrac{c_{12R}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{c_{12R}\Delta_I}{\Delta_R^2 + \Delta_I^2} \\ -\dfrac{c_{21R}\Delta_R}{\Delta_R^2 + \Delta_I^2} & -\dfrac{c_{21R}\Delta_I}{\Delta_R^2 + \Delta_I^2} & z - \dfrac{c_{22R}\Delta_R + c_{22I}\Delta_I}{\Delta_R^2 + \Delta_I^2} & \dfrac{-c_{22R}\Delta_I + c_{22I}\Delta_R}{\Delta_R^2 + \Delta_I^2} \\ \dfrac{c_{21R}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{c_{21R}\Delta_R}{\Delta_R^2 + \Delta_I^2} & \dfrac{c_{22R}\Delta_I - c_{22I}\Delta_R}{\Delta_R^2 + \Delta_I^2} & z - \dfrac{c_{22R}\Delta_R + c_{22I}\Delta_I}{\Delta_R^2 + \Delta_I^2} \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} d_{11R}\Delta_R + d_{11I}\Delta_I & d_{11R}\Delta_I - d_{11I}\Delta_R \\ -d_{11R}\Delta_I + d_{11I}\Delta_R & d_{11R}\Delta_R + d_{11I}\Delta_I \\ d_{21R}\Delta_I & d_{21R}\Delta_I \\ -d_{21R}\Delta_I & d_{21R}\Delta_I \end{bmatrix}$$

$$\tag{683}$$

II) Forward-difference method (Euler)

The state matrix A is described by the relation:

$$A = \begin{bmatrix} 1 - \omega_n T \cdot \gamma \cdot r_s l'_r d & \omega_n T \omega_s & \omega_n T \cdot r_s l_M d & 0 \\ -\omega_n T \cdot \omega_s & 1 - \omega_n T \cdot \gamma \cdot r_s l'_r d & 0 & \omega_n T \cdot r_s l_M d \\ \omega_n T \cdot \gamma \cdot r'_r l_M d & 0 & 1 - \omega_n T \cdot r'_r l_s d & \omega_n T \cdot \omega_r \\ 0 & \omega_n T \cdot \gamma \cdot r'_r l_M d & -\omega_n T \cdot \omega & 1 - \omega_n T \cdot r'_r l_s d \end{bmatrix} \tag{684}$$

The input vector and the corespondent matrix coefficient are:

$$u_s(n) = \begin{bmatrix} U_{sR}(n) \\ U_{sI}(n) \end{bmatrix} \quad B = \begin{bmatrix} \omega_n T & 0 \\ 0 & \omega_n T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{685-686}$$

The other matrix C and D have the same value as for the bilinear transformation case. It results the transfer function matrix as follows:

$$H_\psi(z) = \begin{bmatrix} z - 1 + \omega_n T \cdot \gamma \cdot r_s l'_r d & -\omega_n T \omega_s & -\omega_n T \cdot r_s l_M d & 0 \\ \omega_n T \cdot \omega_s & z - 1 + \omega_n T \cdot \gamma \cdot r_s l'_r d & 0 & -\omega_n T \cdot r_s l_M d \\ -\omega_n T \cdot \gamma \cdot r'_r l_M d & 0 & z - 1 + \omega_n T \cdot r'_r l_s d & -\omega_n T \cdot \omega_r \\ 0 & -\omega_n T \cdot \gamma \cdot r'_r l_M d & \omega_n T \cdot \omega & z - 1 + \omega_n T \cdot r'_r l_s d \end{bmatrix}^{-1} \cdot \begin{bmatrix} \omega_n T & 0 \\ 0 & \omega_n T \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tag{687}$$

III) The backward-difference method

The state matrix $\mathbf{A}$ and coefficients matrix $\mathbf{B}$ can be calculated with the expressions:

$$\mathbf{A} = \frac{1}{\Delta_R^2 + \Delta_I^2} \begin{bmatrix} g_{11R}\Delta_R + g_{11I}\Delta_I & g_{11R}\Delta_I - g_{11I}\Delta_R & g_{12}\Delta_R & g_{12}\Delta_I \\ -g_{11R}\Delta_I + g_{11I}\Delta_R & g_{11R}\Delta_R + g_{11I}\Delta_I & -g_{12}\Delta_I & g_{12}\Delta_R \\ g_{21}\Delta_R & g_{21}\Delta_I & g_{22R}\Delta_R + g_{22I}\Delta_I & g_{22R}\Delta_I - g_{22I}\Delta_R \\ -g_{21}\Delta_I & g_{21}\Delta_R & -g_{22R}\Delta_I + g_{22I}\Delta_R & g_{22R}\Delta_R + g_{22I}\Delta_I \end{bmatrix} \quad (688)$$

where:

$$\mathbf{G} = \begin{bmatrix} g_{11R} + jg_{11I} & g_{12} \\ g_{21} & g_{22R} + g_{22I} \end{bmatrix} = (\Delta_R - j\Delta_I) \cdot \begin{bmatrix} 1 + \omega_n T r'_r l_s d + j\omega_r \omega_n T & \omega_n T r_s l_M d \\ \omega_n T \gamma r'_r l_M d & 1 + \omega_n T \gamma \cdot r_s l'_r d + j\omega_s \omega_n T \end{bmatrix} =$$

$$= \begin{bmatrix} \Delta_R(1 + \omega_n T r'_r l_s d) + \Delta_I \omega_r \omega_n T + j(\Delta_R \omega_r \omega_n T - \Delta_I(1 + \omega_n T r'_r l_s d)) & \Delta_R \omega_n T r_s l_M d - j\Delta_I \omega_n T r_s l_M d \\ \Delta_R \omega_n T \gamma \cdot r'_r l_M d - j\Delta_I \omega_n T \gamma \cdot r'_r l_M d & \Delta_R(1 + \omega_n T \gamma \cdot r_s l'_r d) + \Delta_I \omega_s \omega_n T + j(\Delta_R \omega_s \omega_n T - \Delta_I(1 + \omega_n T \gamma \cdot r_s l'_r d)) \end{bmatrix}$$

$$(689)$$

and:

$$\mathbf{B} = \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot \begin{bmatrix} h_{11R}\Delta_R + h_{11I}\Delta_I & h_{11R}\Delta_I - h_{11I}\Delta_R \\ -h_{11R}\Delta_I + h_{11I}\Delta_R & h_{11R}\Delta_R + h_{11I}\Delta_I \\ h_{21}\Delta_R & h_{21}\Delta_I \\ -h_{21}\Delta_I & h_{21}\Delta_R \end{bmatrix} \quad (690)$$

where:

$$\mathbf{H} = \begin{bmatrix} h_{11R} + jh_{11I} & h_{12} \\ h_{21} & h_{22R} + jh_{22I} \end{bmatrix} = \omega_n T \cdot \begin{bmatrix} 1 + \omega_n T r'_r l_s d + j\omega_r \omega_n T & \omega_n T r_s l_M d \\ \omega_n T \gamma \cdot r'_r l_M d & 1 + \omega_n T \gamma \cdot r_s l'_r d + j\omega_s \omega_n T \end{bmatrix} \quad (691)$$

As the input vector is $\mathbf{u}_s = \begin{bmatrix} U_{sR}(n) \\ U_{sI}(n) \end{bmatrix}$ and the coefficients matrix $\mathbf{C}$ and $\mathbf{D}$ are calculated in a similar way to the previous cases, it results the following transfer function matrix for two inputs (voltage) and four outputs (fluxes):

$$\mathbf{H}_\psi(z) = \begin{bmatrix} z - \dfrac{g_{11R}\Delta_R + g_{11I}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{11R}\Delta_I - g_{11I}\Delta_R}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{12}\Delta_R}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{12}\Delta_I}{\Delta_R^2 + \Delta_I^2} \\ -\dfrac{-g_{11R}\Delta_I + g_{11I}\Delta_R}{\Delta_R^2 + \Delta_I^2} & z - \dfrac{g_{11R}\Delta_R + g_{11I}\Delta_I}{\Delta_R^2 + \Delta_I^2} & \dfrac{g_{12}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{12}\Delta_R}{\Delta_R^2 + \Delta_I^2} \\ -\dfrac{g_{21}\Delta_R}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{21}\Delta_I}{\Delta_R^2 + \Delta_I^2} & z - \dfrac{g_{22R}\Delta_R + g_{22I}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{22R}\Delta_I - g_{22I}\Delta_R}{\Delta_R^2 + \Delta_I^2} \\ \dfrac{g_{21}\Delta_I}{\Delta_R^2 + \Delta_I^2} & -\dfrac{g_{21}\Delta_R}{\Delta_R^2 + \Delta_I^2} & -\dfrac{-g_{22R}\Delta_I + g_{22I}\Delta_R}{\Delta_R^2 + \Delta_I^2} & z - \dfrac{g_{22R}\Delta_R + g_{22I}\Delta_I}{\Delta_R^2 + \Delta_I^2} \end{bmatrix}^{-1} \cdot$$

$$\cdot \frac{1}{\Delta_R^2 + \Delta_I^2} \cdot \begin{bmatrix} h_{11R}\Delta_R + h_{11I}\Delta_I & h_{11R}\Delta_I - h_{11I}\Delta_R \\ -h_{11R}\Delta_I + h_{11I}\Delta_R & h_{11R}\Delta_R + h_{11I}\Delta_I \\ h_{21}\Delta_R & h_{21}\Delta_I \\ -h_{21}\Delta_I & h_{21}\Delta_R \end{bmatrix}$$

$$(692)$$

4.6. Stability analysis

If the resulting discrete-time system is unstable, it can be transformed in a stable one by decreasing the sampling rate. It is possible to select the the sampling rate such that the discrete-time system is always stable, assuming that the pole locations in the continuos s domain are known. However, this presumes an increased computing burden, as there are processed more sampling rates in a time unit. Therefore it is important to analyse the value for the minimum sampling rate and to determine an optimum value as stated by Franklin et al (1997).

The z-domain transfer function is obtained from an analog prototype transfer function by using the following substitutions according to the approximation method:

$s = \dfrac{z-1}{T}$ forward-difference method (Euler)

$s = \dfrac{z-1}{z \cdot T}$ backward-difference method

$s = \dfrac{2}{T} \cdot \dfrac{z-1}{z+1}$ biliniar transformation method (Tustin)

Thus the interdependence between an s-domain pole and the corresponding z-domain pole is:

$z_{\text{pole}} = 1 + s_{\text{pole}}T$ forward-difference method (Euler)

$z_{\text{pole}} = \dfrac{1}{1 - s_{\text{pole}}T}$ backward-difference method

$z_{\text{pole}} = \dfrac{2 + s_{\text{pole}}T}{2 - s_{\text{pole}}T}$ biliniar transformation method (Tustin)

For a stable s-domain pole (i.e., $\alpha < 0$):

$s_{\text{pole}} = \alpha + j\beta$

A z-domain pole is stable if it is located inside the unit circle, i.e., its modulus is less than the unit (one in relative units):

$$\begin{cases} \left| z_{\text{pole}} \right| < 1 \Rightarrow \sqrt{(1+\alpha T)^2 + (\beta T)^2} < 1 \\ 1 + 2\alpha T + \alpha^2 T^2 + \beta^2 T^2 < 1 \Rightarrow T < -\dfrac{2\alpha}{\alpha^2 + \beta^2} \end{cases} \quad \text{forward-difference method (Euler)} \qquad (693)$$

$$\begin{cases} \left| z_{\text{pole}} \right| < 1 \Rightarrow \dfrac{1}{\sqrt{(1-\alpha T)^2 + (\beta T)^2}} < 1 \\ 1 - 2\alpha T + \alpha^2 T^2 + \beta^2 T^2 > 1 \Rightarrow T > \dfrac{2\alpha}{\alpha^2 + \beta^2} \end{cases} \quad \text{backward-difference method} \qquad (694)$$

$$\begin{cases} \left| z_{\text{pole}} \right| < 1 \Rightarrow \sqrt{\dfrac{(2+\alpha T)^2 + (\beta T)^2}{(2-\alpha T)^2 + (\beta T)^2}} < 1 \\ T > 0 \end{cases} \quad \text{biliniar transformation method (Tustin)} \qquad (695)$$

Expressed in terms of the sampling frequency ($f_{\text{sample}} = 1/T$), the above conditions are:

$$f_{\text{sample}} > -\frac{\alpha^2 + \beta^2}{2\alpha} \quad \text{forward-difference method (Euler)} \qquad (696)$$

$$f_{\text{sample}} < \frac{\alpha^2 + \beta^2}{2\alpha} \quad \text{backward-difference method} \qquad (697)$$

$$f_{\text{sample}} > 0 \quad \text{biliniar transformation method (Tustin)} \qquad (698)$$

These expressions prove that the sampling rate can always be selected such that the discrete-time system is stable if the original continuos-time system is stable as well.

The poles position may be variable due to three important factors:

- The simplified assumptions for linearity of the continuous-time model;
- The parameters variation due to environmental effects: temperature, humidity;
- The implementation of the system using fixed-point digital signal processors. The fixed point determines round errors and scaling or quantification errors.

So, even for poles placed inside the unit circle, there is a possibility of unstable operation of the system. A pole location very near to the unit location can be problematic, mainly in environments with short word length of the DSP. Therefore it may be desirable to maximise the distance between the critical pole and the unit circle and set the sampling rate accordingly.

The derivative of the modulus of the pole gives the relations:

$$\frac{d|z_{pole}|}{dT} = \frac{2\alpha + 2\alpha^2 T + 2\beta^2 T}{\sqrt{1 + 2\alpha T + \alpha^2 T^2 + \beta^2 T^2}} \quad \text{forward-difference method (Euler)} \tag{699}$$

$$\frac{d|z_{pole}|}{dT} = \frac{-2\alpha + 2\alpha^2 T + 2\beta^2 T}{\sqrt{(1 - 2\alpha T + \alpha^2 T^2 + \beta^2 T^2)^3}} \quad \text{backward-difference method} \tag{700}$$

$$\frac{d|z_{pole}|}{dT} = \frac{4\left[T^2\alpha(\alpha^2 + \beta^2) - 4T(\alpha^2 + \beta^2) + 4\alpha\right]}{\sqrt{\frac{(2 + \alpha T)^2 + (\beta T)^2}{(2 - \alpha T)^2 + (\beta T)^2}}} \quad \text{biliniar transformation method (Tustin)} \tag{701}$$

By setting the derivative equal to zero it is obtained an expression for the optimum sampling frequency:

$$f_{sample, opt} = -\frac{\alpha^2 + \beta^2}{\beta} \quad \text{forward-difference method (Euler)} \tag{702}$$

$$f_{sample, opt} = \frac{\alpha^2 + \beta^2}{\beta} \quad \text{backward-difference method} \tag{703}$$

$$f_{sample, opt} > 0 \quad \text{biliniar transformation method (Tustin) as the derivative is always positive.} \tag{704}$$

One should notice that the above sampling rate values do not optimise the resemblance between time-domain or frequency-domain responses of the continuos-time and the discrete-time system.

The stable methods (forward-differences, backward-differences and biliniar transformation) map an s-domain point $\alpha + j\beta$ into the z domain as follows:

a) Forward-difference method (Euler)

$$z = \frac{(1 - m)\alpha - j\beta}{\alpha - j\beta} \tag{705}$$

where $m=1$ for the "optimum" sampling rate; $m=2$ for the minimum sampling rate

b) Backward-difference method

$$z = \frac{\beta + j\alpha}{\beta - j(1 - m)\alpha} \tag{706}$$

where $m=1$ for the "optimum" sampling rate; $m=2$ for the minimum sampling rate

c) Biliniar transformation method (Tustin)

$$z = 1 \quad \text{the unit circle,} \tag{707}$$

i.e. the system is stable for any stable s-domain poles ($\alpha < 0$)

5. MATHEMATICAL DISCRETE MODELS FOR THE SINGLE-PHASE INDUCTION MACHINE

5.1. Introduction

Discrete-time computational models have to be derived for an advanced motor control of squirrel-cage type single-phase induction machine. The real-time vector control analysis for the single-phase induction machine can be realised using mathematical discrete models, in a similar way to the three-phase induction machine. Some differences will appear due to the asymmetry of the single-phase induction machine stator configuration.

The same general considerations regarding the discretisation process, valid for the three-phase machine, apply.

The starting point for converting the continuous-time model of the single-phase machine into a discrete one is the system of voltage and flux linkage equations. However, by comparison with the three-phase induction machine discretisation two main differences have to be highlighted:

- The reference system is stationary, fixed to the stator;
- The asymmetrical configuration of the stator windings permits only the analysis in two-axis co-ordinates system. The space vector notation cannot be used.

The complete set of equations for an unsymmetrical single-phase induction machine using the universal mathematical model, with *flux linkage per second units and reactance elements*, is as follows:

Stator voltage equations:

$$u_{qs} = r_m i_{qs} + \frac{1}{\omega_b} \cdot \frac{d}{dt} \lambda_{qs}$$

$$u_{ds} = r_a i_{ds} + \frac{1}{\omega_b} \cdot \frac{d}{dt} \lambda_{ds}$$

(708-709)

Flux linkages equations:

$$\lambda_{ds} = \left(x_{la} + k^2 (1-\gamma_d) \cdot x_m \right) \cdot i_{ds} + \gamma_d \cdot k^2 x_m \cdot \left(i_{ds} + i'_{dR} \right) =$$
$$= \left(x_{la} + k^2 (1-\gamma_d) \cdot x_m \right) \cdot i_{ds} + \lambda'_{d\gamma_d}$$

$$\lambda_{qs} = \left(x_{lm} + (1-\gamma_q) \cdot x_m \right) \cdot i_{qs} + \gamma_q \cdot x_m \cdot \left(i_{qs} + i'_{qR} \right) =$$
$$= \left(x_{lm} + (1-\gamma_q) \cdot x_m \right) \cdot i_{qs} + \lambda'_{q\gamma_q}$$

(710-713)

$$\lambda'_{dR} = \gamma_d \cdot k^2 \left(\gamma_d \cdot x'_r + (\gamma_d - 1) \cdot x_m \right) \cdot i'_{dR} + \gamma_d \cdot k^2 x_m \cdot \left(i_{ds} + i'_{dR} \right) =$$
$$= \lambda'_{d\gamma_d} + \gamma_d \cdot k^2 \left(\gamma_d \cdot x'_r + (\gamma_d - 1) \cdot x_m \right) \cdot i'_{dR}$$

$$\lambda'_{qR} = \gamma_q \cdot \left(\gamma_q \cdot x'_r + (\gamma_q - 1) \cdot x_m \right) \cdot i'_{qR} + \gamma_q \cdot x_m \cdot \left(i_{qs} + i'_{qR} \right) =$$
$$= \lambda'_{q\gamma_q} + \gamma_q \cdot \left(\gamma_q \cdot x_r + (\gamma_q - 1) \cdot x_m \right) \cdot i'_{qR}$$

Rotor voltage equations (cage rotor case):

$$0 = r'_R i'_{dR} + \frac{p}{\omega_b} \lambda'_{dR} + k \cdot \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

$$0 = k^2 r'_R i'_{qR} + \frac{p}{\omega_b} \lambda'_{qR} - \frac{1}{k} \cdot \frac{\omega_r}{\omega_b} \lambda'_{dR}$$

(714-716)

$$r'_R = \gamma_{q(d)}^2 \cdot r'_r$$

The transformed rotor current $i'_{d(q)R}$ equals:

$$i'_{dR} = \frac{i'_{dr}}{\gamma_d}; \quad i'_{qR} = \frac{i'_{qr}}{\gamma_q} \tag{717}$$

The electromagnetic torque equation is:

$$T_e = \frac{P}{2} \cdot \frac{x_m}{\omega_b} \cdot k \cdot (\gamma_d \cdot i_{qs} i'_{dR} - \gamma_q \cdot i_{ds} i'_{qR}) = \frac{P}{2} \cdot \frac{1}{\omega_b} \cdot \left(k\lambda'_{qR} i'_{dR} - \frac{1}{k}\lambda'_{dR} i'_{qR} \right) = \frac{P}{2} \cdot \frac{1}{\omega_b} \left(\frac{1}{k}\lambda_{ds} i_{qs} - k\lambda_{qs} i_{ds} \right) =$$

$$= \frac{P}{2} \cdot \frac{x_m}{k\omega_b \cdot (x_s x'_r - x_m^2)} \cdot \left(\frac{1}{\gamma_d} \cdot \lambda_{qs}\lambda'_{dR} - \frac{1}{\gamma_q} \cdot \lambda_{ds}\lambda'_{qR} \right) \tag{718}$$

If the flux linkages are chosen as independent variables, the currents can be deduced from the expressions:

$$i_{qs} = \frac{\lambda_{qs} x'_r - \frac{1}{\gamma_q} \cdot \lambda'_{qR} x_m}{D}$$

$$i_{ds} = \frac{\lambda_{ds} x'_r - \frac{1}{\gamma_d} \cdot \lambda'_{dR} x_m}{k^2 D} \tag{719-722}$$

$$i'_{qR} = \frac{\frac{1}{\gamma_q} \cdot \lambda'_{qR} x_s - \lambda_{qs} x_m}{\gamma_q D}$$

$$i'_{dR} = \frac{\frac{1}{\gamma_d} \lambda'_{dR} x_s - \lambda_{ds} x_m}{\gamma_d k^2 D}$$

where:

$$x_s = x_m + x_m$$
$$x'_r = x'_{lr} + x_m$$
$$D = x_s x'_r - x_m^2$$

Recapping from the continuous linear mathematical models for the unsymmetrical single-phase induction machine, there are three specific choices of the turn ratio "γ_q" and "γ_d":

a) The rotor flux is selected as reference (inverse Γ-form model):

$$\gamma_q = \frac{x_m}{x_m + x'_r}$$

$$\gamma_d = \frac{k^2 x_m}{k^2 (x_m + x'_r)} = \gamma_q \tag{723}$$

b) The air-gap flux is selected as reference (T-form model):

$$\gamma_q = \gamma_d = 1 \tag{724}$$

c) The stator flux is selected as reference (Γ-form model):

$$\gamma_q = \frac{x_m + x_{lm}}{x_m}$$

$$\gamma_d = \frac{k^2 x_m + x_{la}}{k^2 x_m} \cong \gamma_q \tag{725}$$

If the inverse of D is noted with d, the voltage equations will be re-written in a new form:

$$u_{qs} = r_m \cdot \left(x'_r \, d \cdot \lambda_{qs} - x_m d \cdot \frac{1}{\gamma} \cdot \lambda'_{qR} \right) + \frac{1}{\omega_b} \cdot \frac{d}{dt} \lambda_{qs}$$

$$u_{ds} = \frac{r_a}{k^2} \cdot \left(x'_r \, d \cdot \lambda_{ds} - x_m d \cdot \frac{1}{\gamma} \cdot \lambda'_{dR} \right) + \frac{1}{\omega_b} \cdot \frac{d}{dt} \lambda_{ds}$$

$$0 = r'_R \cdot \left(\frac{1}{\gamma^2} x_s d \cdot \lambda'_{qR} - \frac{1}{\gamma} x_m d \cdot \lambda_{qs} \right) + \frac{1}{\omega_b} \cdot \frac{d}{dt} \lambda'_{qR} - \frac{1}{k} \cdot \frac{\omega_r}{\omega_b} \lambda'_{dR}$$

$$0 = r'_R \cdot \left(\frac{1}{\gamma^2} x_s d \cdot \lambda'_{dR} - \frac{1}{\gamma} x_m d \cdot \lambda_{ds} \right) + \frac{1}{\omega_b} \cdot \frac{d}{dt} \lambda'_{dR} + k \cdot \frac{\omega_r}{\omega_b} \lambda'_{qR}$$

(726-729))

For null initial conditions, if the Laplace transformation is applied, it results a matrix system:

$$\frac{s}{\omega_b} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qR} \\ \lambda'_{dR} \end{bmatrix} = \begin{bmatrix} -r_m x'_r \, d & 0 & r_m x_m d / \gamma & 0 \\ 0 & -r_a x'_r \, d / k^2 & 0 & r_a x_m d / (\gamma \cdot k^2) \\ r'_R x_m d / \gamma & 0 & -r'_R x_s d / \gamma^2 & \omega_r / (k\omega_b) \\ 0 & r'_R x_m d / \gamma & k(\omega_r / \omega_b) & -r'_R x_s d / \gamma^2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qR} \\ \lambda'_{dR} \end{bmatrix} + \begin{bmatrix} u_{qs} \\ u_{ds} \\ 0 \\ 0 \end{bmatrix} \quad (730)$$

or synthetically:

$$sY(s) = A \cdot Y(s) + B \cdot U(s) \tag{731}$$

By comparing to the general form of a state variable system:

$$\begin{cases} s \cdot X(s) = A \cdot X(s) + B \cdot U(s) \\ Y(s) = C \cdot X(s) + D \cdot U(s) \end{cases} \tag{732}$$

It can be observed that the state variable vector is identical to the output vector (in this case the flux linkages per second vector). The inputs vector is the voltages vector (in this case only the stator voltages, as the rotor is short-circuited). By identification, we obtain:

$$A = \omega_b \cdot \begin{bmatrix} -r_m x'_r \, d & 0 & r_m x_m d / \gamma & 0 \\ 0 & -r_a x'_r \, d / k^2 & 0 & r_a x_m d / (\gamma \cdot k^2) \\ r'_R x_m d / \gamma & 0 & -r'_R x_s d / \gamma^2 & \omega_r / (k\omega_b) \\ 0 & r'_R x_m d / \gamma & k(\omega_r / \omega_b) & -r'_R x_s d / \gamma^2 \end{bmatrix} \quad (733\text{-}734)$$

$$B = \omega_b$$

5.2. Bilinear transformation method (Tustin)

Recapping from the mathematical discrete model of three-phase induction machine analysis, the following relation is the bilinear transformation (Tustin) from continuous to discrete domain:

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

The matrix equation from continuous domain becomes in discrete:

$$\frac{2}{T} \cdot \frac{z-1}{z+1} \cdot Y(z) = A \cdot Y(z) + B \cdot U(z) \tag{735}$$

which gives the relation for the inputs vector $Y(z)$:

$$Y(z) = (2f \cdot I - A)^{-1} \cdot (2f \cdot I + A) \cdot z^{-1} \cdot Y(z) + (2f \cdot I - A)^{-1} \cdot B \cdot (1 + z^{-1}) \cdot U(z) \tag{736}$$

where:

$$2f \cdot \boldsymbol{I} \mp \boldsymbol{A} = \begin{bmatrix} 2f \pm \omega_b r_m x'_r d & 0 & \mp \omega_b r_m x_m d / \gamma & 0 \\ 0 & 2f \pm \dfrac{\omega_b r_a x'_r d}{k^2} & 0 & \mp \dfrac{\omega_b r_a x_m d}{k^2 \gamma} \\ \mp \dfrac{\omega_b r'_R x_m d}{\gamma} & 0 & 2f \pm \dfrac{\omega_b r'_R x_s d}{\gamma^2} & \mp \dfrac{\omega_r}{k} \\ 0 & \mp \dfrac{\omega_b r'_R x_m d}{\gamma} & \pm k\omega_r & 2f \pm \dfrac{\omega_b r'_R x_s d}{\gamma^2} \end{bmatrix} \quad (737)$$

The expression of the flux linkages per second in the discrete time domain is obtained when using the bilinear transformation:

$$\begin{bmatrix} \lambda_{qs}(n) \\ \lambda_{ds}(n) \\ \lambda'_{qR}(n) \\ \lambda'_{dR}(n) \end{bmatrix} = \boldsymbol{C} \cdot \begin{bmatrix} \lambda_{qs}(n-1) \\ \lambda_{ds}(n-1) \\ \lambda'_{qR}(n-1) \\ \lambda'_{dR}(n-1) \end{bmatrix} + \boldsymbol{D} \cdot \begin{bmatrix} u_{qs}(n-1)+u_{qs}(n) \\ u_{ds}(n-1)+u_{ds}(n) \\ 0 \\ 0 \end{bmatrix} \quad (738)$$

$$\boldsymbol{C} = (2f - \boldsymbol{A})^{-1} \cdot (2f + \boldsymbol{A})$$
$$\boldsymbol{D} = \omega_b \cdot (2f - \boldsymbol{A})^{-1} \quad (739)$$

The above relations lead to the mathematical discrete model of the single-phase induction machine from Fig. 5.1.

The electromagnetic torque is computed using the stator and rotor flux linkages as independent variables:

$$T_e = \frac{P}{2} \cdot \frac{x_m}{k\omega_b \cdot (x_s x'_r - x_m^2)} \cdot \left(\frac{1}{\gamma_d} \cdot \lambda_{qs} \lambda'_{dR} - \frac{1}{\gamma_q} \cdot \lambda_{ds} \lambda'_{qR} \right) = \frac{P}{2} \cdot \frac{x_m d}{k\omega_b \gamma} \cdot \left(\lambda_{qs} \lambda'_{dR} - \lambda_{ds} \lambda'_{qR} \right) \quad (740)$$

Considering the continuous time domain and using the Laplace transformation, the rotor angular velocity value is readily available from the torque expression:

$$\omega_r = \frac{1}{s} \cdot \left[\frac{P}{2} \cdot \frac{1}{J} (T_e - T_L) \right] \quad (741)$$

The currents vector can be determined according to the following matrix equation:

$$\begin{bmatrix} i_{qs} \\ i_{ds} \\ i'_{qR} \\ i'_{dR} \end{bmatrix} = \begin{bmatrix} x'_r d & 0 & -x_m d / \gamma & 0 \\ 0 & x'_r d / k^2 & 0 & -x_m d /(k^2 \gamma) \\ -x_m d / \gamma & 0 & x_s d / \gamma^2 & 0 \\ 0 & -x_m d /(k^2 \gamma) & 0 & x_s d /(k^2 \gamma^2) \end{bmatrix} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qR} \\ \lambda'_{dR} \end{bmatrix} \quad (742)$$

or in a condensed form:

$$\boldsymbol{i}(n) = \boldsymbol{F} \cdot \boldsymbol{\lambda}(n) \quad (743)$$
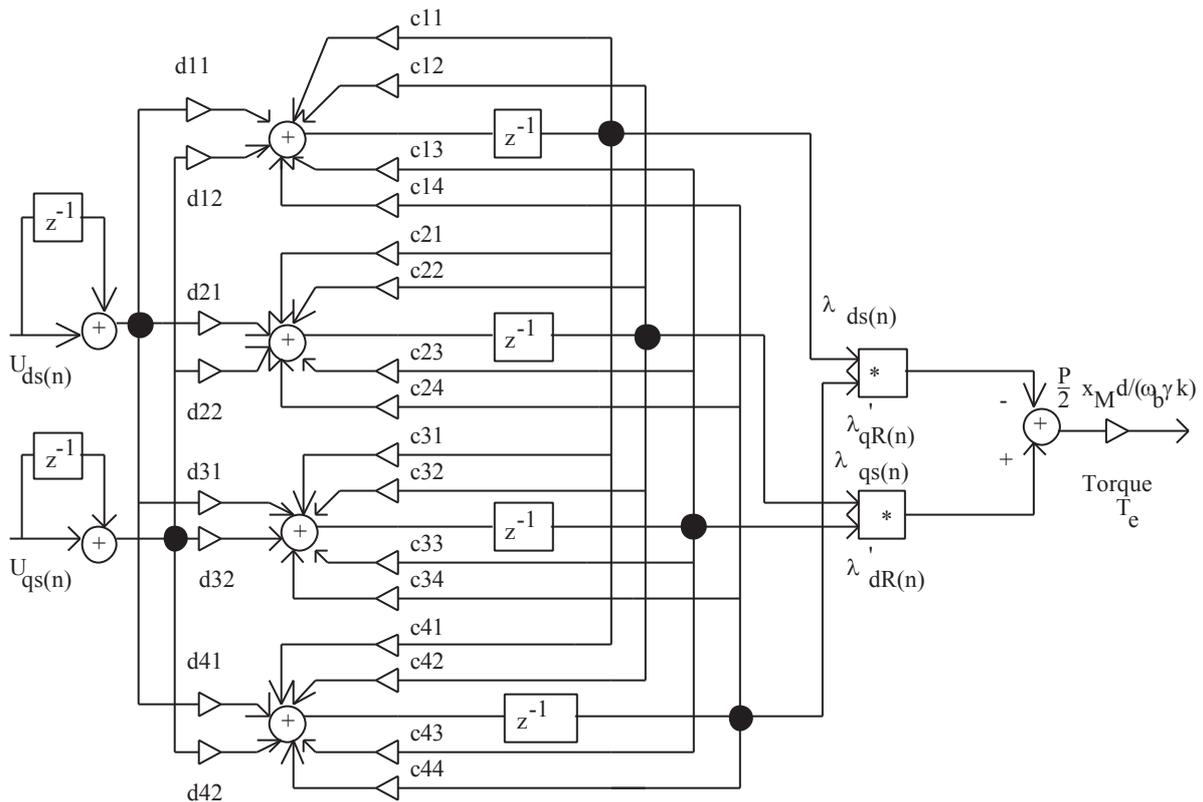
Fig.5.1. Complete discrete single-phase induction machine model based on
bilinear transformation method (Tustin)

The analysis of the block diagram given in Fig. 1 shows that for a complete implementation of the model 27 multipliers, 23 additions operations and 4 delay blocks are necessary. However, this model can be simplified depending on the type of implementation. The general structure of the physical implementation depends on the placement of the blocks and the system frequency response will be modified accordingly.

5.3. Forward-differences method (Euler)

The system will become unstable if the sampling period for a discrete model is incorrectly chosen. The transformation from the continuous-time to the discrete-time domain can be made using the Euler method, which by definition is:

$$\frac{dx}{dt} = \frac{x(n+1) - x(n)}{T} \tag{744}$$

where T is the sampling period.

The voltage equations for the single-phase induction machine with cage rotor are transformed as follows:

$$u_{qs}(n) = r_m \cdot \left(x'_r \, d \cdot \lambda_{qs}(n) - x_m d \cdot \frac{1}{\gamma} \lambda'_{qR}(n) \right) + \frac{1}{\omega_b} \cdot \frac{\lambda_{qs}(n+1) - \lambda_{qs}(n)}{T}$$

$$u_{ds}(n) = \frac{r_a}{k^2} \cdot \left(x'_r \, d \cdot \lambda_{ds}(n) - x_m d \cdot \frac{1}{\gamma} \lambda'_{dR}(n) \right) + \frac{1}{\omega_b} \cdot \frac{\lambda_{ds}(n+1) - \lambda_{ds}(n)}{T}$$

$$0 = r'_R \cdot \left(\frac{1}{\gamma^2} x_s d \cdot \lambda'_{qR}(n) - \frac{1}{\gamma} x_m d \cdot \lambda_{qs}(n) \right) + \frac{1}{\omega_b} \cdot \frac{\lambda'_{qR}(n+1) - \lambda'_{qR}(n)}{T} - \frac{1}{k} \cdot \frac{\omega_r}{\omega_b} \lambda'_{dR}(n)$$

$$0 = r'_R \cdot \left(\frac{1}{\gamma^2} x_s d \cdot \lambda'_{dR}(n) - \frac{1}{\gamma} x_m d \cdot \lambda_{ds}(n) \right) + \frac{1}{\omega_b} \cdot \frac{\lambda'_{dR}(n+1) - \lambda'_{dR}(n)}{T} + k \cdot \frac{\omega_r}{\omega_b} \lambda'_{qR}(n)$$

(745-748)

which gives the matrix relation for the flux linkages per second in discrete representation:

$$\begin{bmatrix} \lambda_{qs}(n+1) \\ \lambda_{ds}(n+1) \\ \lambda'_{qR}(n+1) \\ \lambda'_{dR}(n+1) \end{bmatrix} = \begin{bmatrix} -r_m x'_r \, d\omega_b T + 1 & 0 & r_m x_m d\omega_b T/\gamma & 0 \\ 0 & -r_a x'_r \, d\omega_b T/k^2 + 1 & 0 & r_a x_m d\omega_b T/(k^2\gamma) \\ r'_R x_m d\omega_b T/\gamma & 0 & -r'_R x_s d\omega_b T/\gamma^2 + 1 & \omega_r T/k \\ 0 & r'_R x_m d\omega_b T/\gamma & -k\omega_r T & -r'_R x_s d\omega_b T/\gamma^2 + 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda_{qs}(n) \\ \lambda_{ds}(n) \\ \lambda'_{qR}(n) \\ \lambda'_{dR}(n) \end{bmatrix} + \omega_b T \cdot \begin{bmatrix} u_{qs}(n) \\ u_{ds}(n) \\ 0 \\ 0 \end{bmatrix}$$

(749)

or in a condensed form:
$$\boldsymbol{\lambda}(n+1) = \boldsymbol{E} \cdot \boldsymbol{\lambda}(n) + \omega_b T \cdot \boldsymbol{u}(n) \qquad (750)$$

As the flux linkages per second vector is the independent variables vector, the currents are determined according to the following matrix equation:

$$\begin{bmatrix} i_{qs} \\ i_{ds} \\ i'_{qR} \\ i'_{dR} \end{bmatrix} = \begin{bmatrix} x'_r \, d & 0 & -x_m d/\gamma & 0 \\ 0 & x'_r \, d/k^2 & 0 & -x_m d/(k^2\gamma) \\ -x_m d/\gamma & 0 & x_s d/\gamma^2 & 0 \\ 0 & -x_m d/(k^2\gamma) & 0 & x_s d/(k^2\gamma^2) \end{bmatrix} \cdot \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda'_{qR} \\ \lambda'_{dR} \end{bmatrix} \qquad (751)$$

or in a condensed form:
$$\boldsymbol{i}(n) = \boldsymbol{F} \cdot \boldsymbol{\lambda}(n) \qquad (752)$$

Similar to the bilinear transformation case, the electromagnetic torque is computed with the relation:

$$T_e = \frac{P}{2} \cdot \frac{x_m}{k\omega_b \cdot (x_s x'_r - x_m^2)} \cdot \left(\frac{1}{\gamma_d} \cdot \lambda_{qs} \lambda'_{dR} - \frac{1}{\gamma_q} \cdot \lambda_{ds} \lambda'_{qR} \right) = \frac{P}{2} \cdot \frac{x_m d}{k\omega_b \gamma} \cdot \left(\lambda_{qs} \lambda'_{dR} - \lambda_{ds} \lambda'_{qR} \right) \qquad (753)$$

For the forward-differences method (Euler), the mathematical discrete model of the single-phase induction machine is described by the block diagram in Fig. 5.2. This model is characterised by the following number of blocks requested for implementation: 9 additions, 15 multipliers and 6 delay blocks. By comparison with the previous discretisation method, it is obviously the simpler implementation structure for the forward-difference method. The computational burden is 1.8 times greater with the bilinear transformation than it is with the forward-difference method.
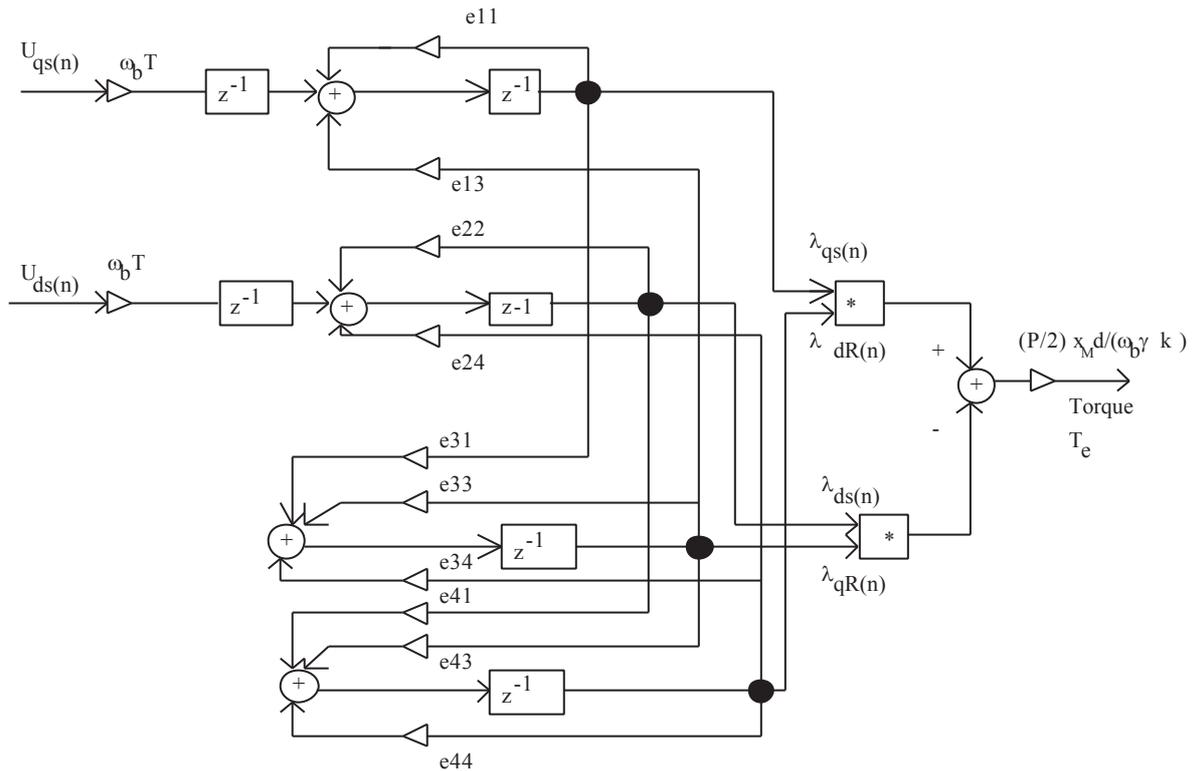
Fig.5.2. Complete discrete single-phase induction machine model based on
forward-difference method (Euler)

5.4. Backward-differences method

This simply to apply discretisation method allows the implementation of a stable discrete system if the analogue version in continuous time domain is also stable. Using this method, even unstable continuous models can be transformed in discrete stable systems. Nevertheless, it must be stated that due to the distortions in the frequency response of the system, a lower sampling period has to be used. The flux linkages per second vector can be obtained by applying the relation used for the same transformation type from the three-phase induction machine to the single-phase induction machine:

$$\frac{1-z^{-1}}{T} \cdot \mathbf{Y}(z) = \mathbf{A} \cdot \mathbf{Y}(z) + \mathbf{B} \cdot \mathbf{U}(z) \Rightarrow \left(\mathbf{I} - \mathbf{A} \cdot T \right) \cdot \mathbf{Y}(z) = z^{-1} \cdot \mathbf{Y}(z) + \mathbf{B} \cdot T \cdot \mathbf{U}(z) \tag{754}$$

or in a matrix form:

$$\begin{bmatrix} \lambda_{qs}(n) \\ \lambda_{ds}(n) \\ \lambda'_{qR}(n) \\ \lambda'_{dR}(n) \end{bmatrix} = \left(\mathbf{I} - \mathbf{A} \cdot T \right)^{-1} \cdot \begin{bmatrix} \lambda_{qs}(n-1) \\ \lambda_{ds}(n-1) \\ \lambda'_{qR}(n-1) \\ \lambda'_{dR}(n-1) \end{bmatrix} + \left(\mathbf{I} - \mathbf{A} \cdot T \right)^{-1} \cdot \omega_b T \cdot \begin{bmatrix} u_{qs}(n) \\ u_{ds}(n) \\ 0 \\ 0 \end{bmatrix} \tag{755}$$

or in a condensed form:

$$\boldsymbol{\lambda}(n) = \mathbf{G} \cdot \boldsymbol{\lambda}(n-1) + \mathbf{H} \cdot \mathbf{u}(n) \tag{756}$$

where:

$$
\boldsymbol{G} = \left(\boldsymbol{I} - \boldsymbol{A} \cdot T \right)^{-1} = \begin{bmatrix} 1 + \omega_b Tr_m x'_r d & 0 & -\dfrac{\omega_b Tr_m x_m d}{\gamma} & 0 \\[2mm] 0 & 1 + \dfrac{\omega_b Tr_a x'_r d}{k^2} & 0 & -\dfrac{\omega_b Tr_a x_m d}{k^2 \gamma} \\[2mm] -\dfrac{\omega_b Tr'_r x_m d}{\gamma} & 0 & 1 + \dfrac{\omega_b Tr'_r x_s d}{\gamma^2} & -\dfrac{\omega_r T}{k} \\[2mm] 0 & -\dfrac{\omega_b Tr'_r x_m d}{\gamma} & k \omega_r T & 1 + \dfrac{\omega_b Tr'_r x_s d}{\gamma^2} \end{bmatrix}^{-1} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}
$$

$$ \tag{757} $$

and:

$$ \boldsymbol{H} = \boldsymbol{G} \cdot \omega_b T \tag{758} $$

The matrix elements g_{ij} , i,j = 1, 2, 3, 4 expressions are not detailed here, due to the space limitations.

Like in the previous cases, the electromagnetic torque represents the output of the discrete system:

$$ T_e = \frac{P}{2} \cdot \frac{x_m}{k\omega_b \cdot \left(x_s x'_r - x_m^2 \right)} \cdot \left(\frac{1}{\gamma_d} \cdot \lambda_{qs} \lambda'_{dR} - \frac{1}{\gamma_q} \cdot \lambda_{ds} \lambda'_{qR} \right) = \frac{P}{2} \cdot \frac{x_m d}{k\omega_b \gamma} \cdot \left(\lambda_{qs} \lambda'_{dR} - \lambda_{ds} \lambda'_{qR} \right) \tag{759} $$

The discrete mathematical model obtained through the backward transformation method is illustrated in Fig. 5.3. A comparative computational burden for different methods of discretisation is given in Table 5.I.

TABLE 5.I

Method \ Real operation	Additions	Multiplications	Delays
Forward-difference method (Euler)	9	15	6
Backward-difference method	21	27	4
Bilinear transformation method (Tustin)	23	27	4

The following conclusions can be drawn regarding different implementation options of a discrete mathematical model for the single-phase induction machine:
- the pulse invariance method and backward-difference method determine similar discrete mathematical model for the machine;
- the step invariance method and the forward-difference method determine similar mathematical discrete model for the machine.
- the frequency characteristic of the model differs from one method to another.
- the forward-difference method (Euler) presents the minimum computing time.
- when accuracy is the determining factor, the bilinear transformation method (Tustin) has to be chosen for the implementation of the mathematical discrete model of the machine.
- it is also possible to apply a hybrid approach where the stator and rotor equations are discretised using different transformation methods.
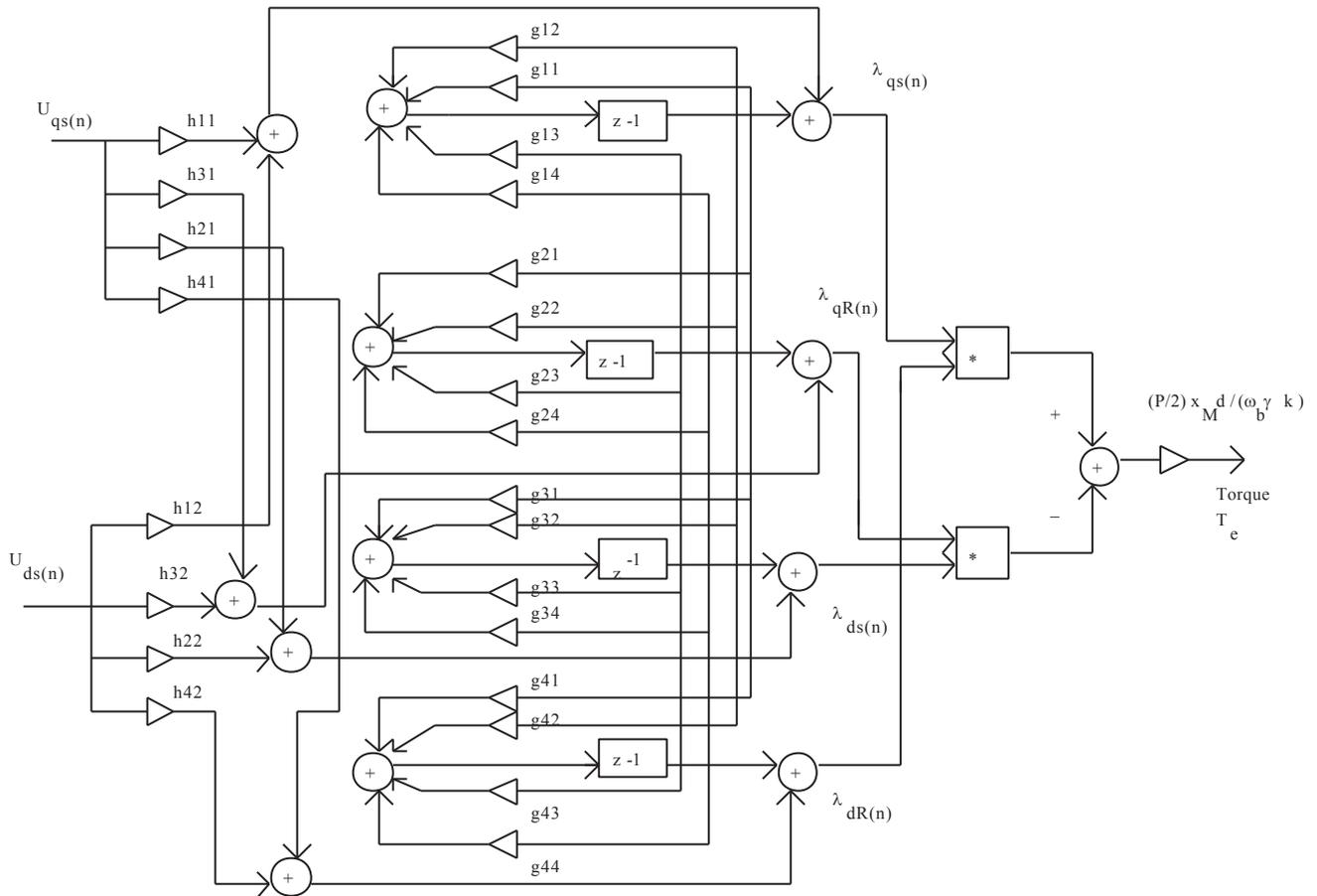
Fig. 5.3. Complete discrete single-phase induction machine model based on backward-difference method

5.5. Z-domain transfer functions

Recapping some general considerations of linear algebra, a state-variable system using matrix notation in z-domain is described as:

$$x_s(n+1) = A \cdot x_s(n) + B \cdot u_s(n)$$
$$y_s(n) = C \cdot x_s(n) + D \cdot u_s(n) \tag{760}$$
$$x_s(n) = \left[x_1(n), \ ..., \ x_m(n) \right]^{\mathrm{T}}$$

where $x_s(n)$ is the vector for the state variables, $u_s(n)$ and $y_s(n)$ are the input and the output vectors respectively, and A is the state matrix. The impulse response sequence in terms of the state-variable description is given by the relation:

$$h(k) = \begin{cases} D, & \text{for} \ \ k = 0 \\ C \cdot A^{k-1}, & \text{for} \ \ k > 0 \end{cases} \tag{761}$$

and the transfer function matrix:

$$H_\lambda(z) = \sum_{k=0}^{\infty} h(k) \cdot z^{-k} = D + C \cdot (zI - A)^{-1} \cdot B \tag{762}$$

For the analysed system, i.e. single-phase induction machine, there are four real outputs (the stator and rotor flux linkages per second) and two real inputs (stator voltages). Also, the state variables are the outputs directly. We can establish the notation:

$$x_s(n) = y_s(n) = \begin{bmatrix} \lambda_{qs}(n) \\ \lambda_{ds}(n) \\ \lambda'_{qR}(n) \\ \lambda'_{dR}(n) \end{bmatrix} \tag{763}$$

The state matrix A, the input vector u_s and the coefficients matrix B, C, and D have different elements according to the transformation method used for implementing the discrete mathematical model for the single-phase induction machine.

I) Bilinear transformation method (Tustin):

By comparing the already established relations with the general form of a state-variable description, for this transformation method we have the notation:

The inputs vector:

$$u_s(n) = \begin{bmatrix} u_{qs}(n) + u_{qs}(n-1) \\ u_{ds}(n) + u_{ds}(n-1) \\ 0 \\ 0 \end{bmatrix} \tag{764}$$

The state matrix:

$$A = \begin{bmatrix} 2f + \omega_b r_m x'_r d & 0 & -\omega_b r_m x_m d / \gamma & 0 \\ 0 & 2f + \dfrac{\omega_b r_a x'_r d}{k^2} & 0 & -\dfrac{\omega_b r_a x_m d}{k^2 \gamma} \\ -\dfrac{\omega_b r'_R x_m d}{\gamma} & 0 & 2f + \dfrac{\omega_b r'_R x_s d}{\gamma^2} & \dfrac{-\omega_r}{k} \\ 0 & -\dfrac{\omega_b r'_R x_m d}{\gamma} & +k\omega_b & 2f + \dfrac{\omega_b r'_R x_s d}{\gamma^2} \end{bmatrix}^{-1} \cdot$$

$$\begin{bmatrix} 2f - \omega_b r_m x'_r d & 0 & +\omega_b r_m x_m d / \gamma & 0 \\ 0 & 2f - \dfrac{\omega_b r_a x'_r d}{k^2} & 0 & +\dfrac{\omega_b r_a x_m d}{k^2 \gamma} \\ +\dfrac{\omega_b r'_R x_m d}{\gamma} & 0 & 2f - \dfrac{\omega_b r'_R x_s d}{\gamma^2} & +\dfrac{\omega_r}{k} \\ 0 & +\dfrac{\omega_b r'_R x_m d}{\gamma} & -k\omega_r & 2f - \dfrac{\omega_b r'_R x_s d}{\gamma^2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{765}$$

The coefficients matrix is:

$$B = \omega_b \cdot \begin{bmatrix} 2f + \omega_b r_m x'_r d & 0 & -\omega_b r_m x_m d / \gamma & 0 \\ 0 & 2f + \dfrac{\omega_b r_a x'_r d}{k^2} & 0 & -\dfrac{\omega_b r_a x_m d}{k^2 \gamma} \\ -\dfrac{\omega_b r'_R x_m d}{\gamma} & 0 & 2f + \dfrac{\omega_b r'_R x_s d}{\gamma^2} & \dfrac{-\omega_r}{k} \\ 0 & -\dfrac{\omega_b r'_R x_m d}{\gamma} & +k\omega_r & 2f + \dfrac{\omega_b r'_R x_s d}{\gamma^2} \end{bmatrix}^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \tag{766}$$

The elements of the matrix A and B require too much space to be detailed here.

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{767}
$$

We obtain the transfer function described by the 4 rows, 4 columns non-zero elements matrix:

$$
\boldsymbol{H}_\lambda(z) = \begin{bmatrix} z-a_{11} & -a_{12} & -a_{13} & -a_{4} \\ -a_{21} & z-a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & z-a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & z-a_{44} \end{bmatrix}^{-1} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} h_{11}(z) & h_{12}(z) & h_{13}(z) & h_{14}(z) \\ h_{21}(z) & h_{22}(z) & h_{23}(z) & h_{24}(z) \\ h_{31}(z) & h_{32}(z) & h_{33}(z) & h_{34}(z) \\ h_{41}(z) & h_{42}(z) & h_{43}(z) & h_{44}(z) \end{bmatrix} \tag{768}
$$

where the elements h_{ij} of the transfer function matrix may be computed through linear algebraic computation.

The transfer function for the currents vector considered as output in relation to the voltages vector as input can be determined as well:

$$
\boldsymbol{H}_i(z) = d \cdot \begin{bmatrix} x'_r h_{11}(z) - \dfrac{x_m}{\gamma} h_{31}(z) & x'_r h_{12}(z) - \dfrac{x_m}{\gamma} h_{32}(z) \\[3mm] \dfrac{x'_r}{k^2} h_{21}(z) - \dfrac{x_m}{k^2\gamma} h_{41}(z) & \dfrac{x'_r}{k^2} h_{22}(z) - \dfrac{x_m}{k^2\gamma} h_{42}(z) \\[3mm] \dfrac{x_s}{\gamma^2} h_{31}(z) - \dfrac{x_m}{\gamma} h_{11}(z) & \dfrac{x_s}{\gamma^2} h_{32}(z) - \dfrac{x_m}{\gamma} h_{12}(z) \\[3mm] \dfrac{x_s}{(k\gamma)^2} h_{41}(z) - \dfrac{x_m}{k^2\gamma} h_{21}(z) & \dfrac{x_s}{(k\gamma)^2} h_{42}(z) - \dfrac{x_m}{k^2\gamma} h_{22}(z) \end{bmatrix} \tag{769}
$$

The corresponding input vector will be:

$$
\boldsymbol{u}_s = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{where:} \quad \begin{cases} u_1 = u_{qs} \cdot \left(1+z^{-1}\right) \\ u_2 = u_{ds} \cdot \left(1+z^{-1}\right) \end{cases}
$$

II) Forward-differences method (Euler)

The state matrix A is described by the relation:

$$
A = \begin{bmatrix} -r_m x'_r d\omega_b T +1 & 0 & r_m x_m d\omega_b T/\gamma & 0 \\ 0 & -r_a x'_r d\omega_b T/k^2 +1 & 0 & r_a x_m d\omega_b T/(k^2\gamma) \\ r'_R x_m d\omega_b T/\gamma & 0 & -r'_R x_s d\omega_b T/\gamma^2 +1 & \omega_r T/k \\ 0 & r'_R x_m d\omega_b T/\gamma & -k\omega_r T & -r'_R x_s d\omega_b T/\gamma^2 +1 \end{bmatrix} \tag{770}
$$

The input vector and the corespondent matrix coefficients are:

$$
\boldsymbol{u}_s(n) = \begin{bmatrix} u_{qs}(n) \\ u_{ds}(n) \end{bmatrix} \quad B = \begin{bmatrix} \omega_b T & 0 \\ 0 & \omega_b T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{771-772}
$$

The other matrix C and D have the same value as for the bilinear transformation case. The transfer function matrix results as follows:

$$H_{\lambda}(z) = \begin{bmatrix} z + r_{m}x'_{r}\,d\omega_{b}T - 1 & 0 & -r_{m}x_{m}d\omega_{b}T/\gamma & 0 \\ 0 & z + r_{a}x'_{r}\,d\omega_{b}T/k^{2} - 1 & 0 & -r_{a}x_{m}d\omega_{b}T/(k^{2}\gamma) \\ -r'_{R}x_{m}d\omega_{b}T/\gamma & 0 & z + r'_{R}x_{s}d\omega_{b}T/\gamma^{2} - 1 & -\omega_{r}T/k \\ 0 & -r'_{R}x_{m}d\omega_{b}T/\gamma & k\omega_{r}T & z + r'_{R}x_{s}d\omega_{b}T/\gamma^{2} - 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \omega_{b}T & 0 \\ 0 & \omega_{b}T \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(773)

A detailed expression for each element of the matrix H_{λ} (z) with 4 rows, 2 columns of non-zero elements, is beyond the scope of this work. The transfer function valid for considering currents vector as outputs, when voltages vector represents the inputs, is identical in symbolic form with the bilinear transformation method case.

III) Backward-differences method

Similar to the cases previously analysed, the state matrix A consists the main computational burden when determining the transfer function matrix H_{λ}. By comparing the established relation for the discrete domain model through this method, we get the following notation:

$$A = \begin{bmatrix} 1 + \omega_{b}Tr_{m}x'_{r}\,d & 0 & -\dfrac{\omega_{b}Tr_{m}x_{m}d}{\gamma} & 0 \\ 0 & 1 + \dfrac{\omega_{b}Tr_{a}x'_{r}\,d}{k^{2}} & 0 & -\dfrac{\omega_{b}Tr_{a}x_{m}d}{k^{2}\gamma} \\ -\dfrac{\omega_{b}Tr'_{r}x_{m}d}{\gamma} & 0 & 1 + \dfrac{\omega_{b}Tr'_{r}x_{s}d}{\gamma^{2}} & -\dfrac{\omega_{r}T}{k} \\ 0 & -\dfrac{\omega_{b}Tr'_{r}x_{m}d}{\gamma} & k\omega_{r}T & 1 + \dfrac{\omega_{b}Tr'_{r}x_{s}d}{\gamma^{2}} \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

(774)

and the input vector u_{s} and the coefficients matrix B are:

$$u_{s}(n) = \begin{bmatrix} u_{qs}(n) \\ u_{ds}(n) \\ 0 \\ 0 \end{bmatrix} \qquad B = A \cdot \omega_{b}T$$

(775-776)

The matrix C and D are identical to that ones previously determined.

We obtain the transfer function matrix H_{λ} as 4 rows, 4 columns matrix with non zero elements:

$$H_{\lambda}(z) = \omega_{b}T \cdot \begin{bmatrix} z - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & z - a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & z - a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & -a_{44} \end{bmatrix}^{-1} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

(777)

The complete expressions for computing the transfer function matrix elements require intense linear algebra computation, and therefore it is recommended to determine these values according to the concrete machine parameters. All the considerations made for the currents vector input case are also valid.

6. LINEARISATION OF THE INDUCTION MACHINE MATHEMATICAL MODEL

6.1. Introduction

An important problem related to the modelling of the induction machine is the non-linearity of the equations that describe its operation. This phenomenon appears in the voltage equations and the electromagnetic torque relation as well, due to the products between the state variables. When a control system with induction machine is designed, it is very useful to linearise the machine equations.

Basically, the linearised equations are obtainable in two ways (Krause et al - 1995), (Rogers - 1965). Firstly, the most used method is the Taylor series expansion of a particular variable (for example voltage, current, flux linkage, or torque) around the steady-state operating point and then by neglecting all second-order terms. Alternatively, it is possible to obtain the linearised equations by expressing all the variables as the sum of their values in the operating point and their incremental value, by neglecting the terms which comprise products of incremental values and by eliminating the steady-state terms. This is called the small-signal form of the machine equations. The result is a differential linearised equations set which describes the dynamic behaviour of the machine when small displacements from the operation point are present. The induction machine can be analysed in this way as a linear system, and it is possible to apply the basic linear system theory in order to compute the eigenvalues and to establish transfer functions for use in the design of controls for these machines.

By definition, the initial displacements of the state-variables are considered to be zero. That is, any machine variable can be written as:

$$x = x_0 + \Delta x \tag{778}$$

where x_0 is the value for the variable x in the fixed-operation point, and Δx is an incremental value from this value.

The equations set for the state-variables are:

$$p\boldsymbol{x}(t) = \boldsymbol{A} \cdot \boldsymbol{x}(t) + \boldsymbol{B} \cdot \boldsymbol{u}(t) + \boldsymbol{E} \cdot \boldsymbol{z}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C} \cdot \boldsymbol{x}(t) + \boldsymbol{D} \cdot \boldsymbol{u}(t) \tag{779-780}$$

where $\boldsymbol{x}$ is the state-variables vector, $\boldsymbol{u}$ is the inputs vector, $\boldsymbol{z}$ is the perturbations vector, $\boldsymbol{y}$ is the outputs vector, $\boldsymbol{A}$ the state system matrix, $\boldsymbol{B}$ the inputs matrix, $\boldsymbol{C}$ the outputs matrix, $\boldsymbol{D}$ the input-output matrix, $\boldsymbol{E}$ the perturbations matrix.

If $\boldsymbol{u}$ is set to zero the general solution of the homogenous or force-free linear differential equation becomes:

$$\boldsymbol{x} = e^{\boldsymbol{A}t} \boldsymbol{K} \tag{781}$$

where $\boldsymbol{K}$ is a vector formed by an arbitrary set of initial conditions. The exponential $e^{\boldsymbol{A}t}$ represents the unforced response of the system. It is defined as the state transition matrix. Small-signal stability is assured if all elements of the transition matrix approach zero asymptotically as time approaches infinity. For the stability analysis, the characteristic equation of A, defined as follows, is used:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \tag{782}$$

In the previous equation $\boldsymbol{I}$ is the identity matrix and λ are the roots of the characteristic equation, referred as eigenvalues, characteristic roots or latent roots.

The eigenvalues provide a simple means of predicting the behaviour of an induction machine at any balanced operating conditions. For a real eigenvalue, the induction machine has an exponential response, and signifies a movement away from the operating point. A real eigenvalue is positive

over the positive-slope region of the torque-speed curve, and becomes negative after maximum steady-state torque. When the eigenvalues are complex they occur as conjugate pairs and signify a mode of oscillation of the state variables. Negative real parts correspond to oscillations, which decrease exponentially with time, meaning a stable condition, while positive real parts correspond to an exponential increase with time, an unstable condition.

Usually, the starting point for the machine system analysis is from the voltage equations, combined with the mechanical equation. So, if the input vector is formed by the stator and rotor voltages plus the load torque, the matrix equation of the system is written as follows:

$$\boldsymbol{u} = \boldsymbol{L} \cdot p\boldsymbol{x} + \boldsymbol{R} \cdot \boldsymbol{x} = \left[\boldsymbol{L}p + \boldsymbol{R} \right] \cdot \boldsymbol{x} \tag{783}$$

where $\left[\boldsymbol{L}p + \boldsymbol{R} \right]$ is denoted as the motional impedance matrix of the machine.

It is possible to obtain the eigenvalues by substituting the differential operator p with the roots symbol λ, thus the eigen-motional impedance matrix $\boldsymbol{Z}_m$ can be formed, and the characteristic equation will be:

$$\det\left[\boldsymbol{Z}_m(\lambda) \right] = 0 \tag{784}$$

The design and analysis of controls associated with machines (i.e. vector control) require the transfer function of the actual electrical machines, viewed as a system. Using the previous state-variable set of equations, and substituting the differential operator p with the Laplace operator s, the input-output transfer function can be expressed as follows, considering no perturbations ($z = 0$):

$$\boldsymbol{H}(s) = \frac{\boldsymbol{y}(s)}{\boldsymbol{u}(s)} = \boldsymbol{C} \cdot \left[s\boldsymbol{I} - \boldsymbol{A} \right]^{-1} \cdot \boldsymbol{B} + \boldsymbol{D} \tag{785}$$

If the inputs vector is zero, the output-perturbation transfer function can be computed as:

$$\boldsymbol{H}_z(s) = \frac{\boldsymbol{y}(s)}{\boldsymbol{z}(s)} = \boldsymbol{C} \cdot \left[s\boldsymbol{I} - \boldsymbol{A} \right]^{-1} \cdot \boldsymbol{E} \tag{786}$$

6.2. Three-phase induction machine linearisation

In the concrete case of the symmetrical three-phase induction machine, the per unit version is selected for compactness, and the input vector $\boldsymbol{u}$ is chosen to be the voltages vector. Practically, it contains two terms for the stator voltages, and two terms for the rotor voltages expressed in two-axis coordinate system. However, a single input variables or a linear combination of several input variables can also be selected. In this formulation, we can express $\boldsymbol{u}$ as:

$$\boldsymbol{u} = \boldsymbol{G} \cdot \Delta u_i \tag{787}$$

where $\boldsymbol{G}$ is a column matrix and Δu_i is an input variable such as a linear combination of several input variables i.e. the amplitude of the terminal voltage.

The perturbations vector is usually given by the variation of the load torque:

$$z = \Delta T_L \tag{788}$$

Recapping from the d-q modelling of the three-phase induction machine, there are fourteen possible sets of state-variables: currents, flux linkages or mixed flux linkages and currents pairs. Let the selected pair of incremental state-space variables be denoted as $\Delta \underline{x}_1, \Delta \underline{x}_2$ and detailed as:

$$\begin{aligned} \Delta \underline{x}_1 &= \Delta x_{1d} + j\Delta x_{1q} \\ \Delta \underline{x}_2 &= \Delta x_{2d} + j\Delta x_{2q} \end{aligned} \tag{789}$$

The small-signal expression for electromagnetic torque can be deduced from the large-signal torque expression:

$$T_e = K \cdot \left(x_{1q} x_{2d} - x_{1d} x_{2q} \right) \tag{790}$$

where K denotes a constant corresponding to the selected set of state-variables (i.e. $K = \dfrac{x_m}{D}$ for flux linkages as state-variables).

If we apply the small-displacement approach to the state-variables, it results:

$$\Delta T_e = K \cdot \left(x_{1q0} \Delta x_{2d} + x_{2d0} \Delta x_{1q} - x_{1d0} \Delta x_{2q} - x_{2q0} \Delta x_{1d} \right) \tag{791}$$

For a complete description of the linearised induction machine model, we have to include also the mechanical equation, expressed in small-signal form:

$$p \frac{\Delta \omega_r}{\omega_b} = \frac{1}{2H} \cdot \left(\Delta T_e - \Delta T_L \right) \tag{791}$$

The starting point for the linearised state-variable models is given by voltage equations system written in *stationary reference frame*:

$$u = L \cdot px + R \cdot x \tag{792}$$

where: x is the selected set of state-variables and represent also the output of the model, u is the previously detailed input vector, L is the coefficients matrix (it can be formed by reactance values, or non-dimensional elements) for multiplying the time derivative of the state-variables, R is the coefficients matrix (it can be formed by resistances and reactances values or non-dimensional elements) for multiplying the state-variables.

Finally, the elements for the three-phase induction machine mathematical model in linearised form will be:

$$\begin{aligned} px(t) &= A \cdot x(t) + B \cdot u(t) + E \cdot z(t) \\ y(t) &= C \cdot x(t) + D \cdot u(t) \end{aligned} \tag{793-794}$$

where:

$$x(t) = \begin{bmatrix} \Delta x_{1q} & \Delta x_{1d} & \Delta x_{2q} & \Delta x_{2d} & \Delta \omega_r \end{bmatrix}^T \tag{795}$$

$$u(t) = \begin{bmatrix} \Delta u_{qs} & \Delta u_{ds} & \Delta u_{qr} & \Delta u_{dr} & \Delta T_L \end{bmatrix}^T \tag{796}$$

and:

$$A = -M^{-1} \cdot S$$

$$B = M^{-1}$$

$$M = \begin{bmatrix} L & 0(4,1) \\ l_{21} & -\dfrac{1}{2H} \end{bmatrix}; \quad S = \begin{bmatrix} R & r_{12} \\ 0(1,4) & 0 \end{bmatrix} \tag{797-798}$$

$$l_{21} = K \cdot \begin{bmatrix} x_{2d0} & -x_{2q0} & -x_{1d0} & x_{1q0} \end{bmatrix}$$

$$r_{12} = \begin{bmatrix} 0 & 0 & \lambda_{dr0}(x_{1d0}, x_{2d0}) & -\lambda_{qr0}(x_{1q0}, x_{2q0}) \end{bmatrix}^T ;$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5 \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 0_5 \tag{799-800}$$

The perturbations vector can be extracted from the inputs vector as:

$$z(t) = \Delta T_L \tag{801}$$

and:

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & -\dfrac{1}{2H} \end{bmatrix}^{\mathrm{T}} \tag{802}$$

The eigenvalues can be determined from the characteristic equation. The other way of determining the roots of this equation is by using the motional matrix impedance:

$$\det\big[Z_m(\lambda) \big] = \det\big[M\lambda + S \big] = 0 \tag{803}$$

In Fig. 6.1 is illustrated the small-signal equivalent circuit of the three-phase induction machine in stationary reference frame:
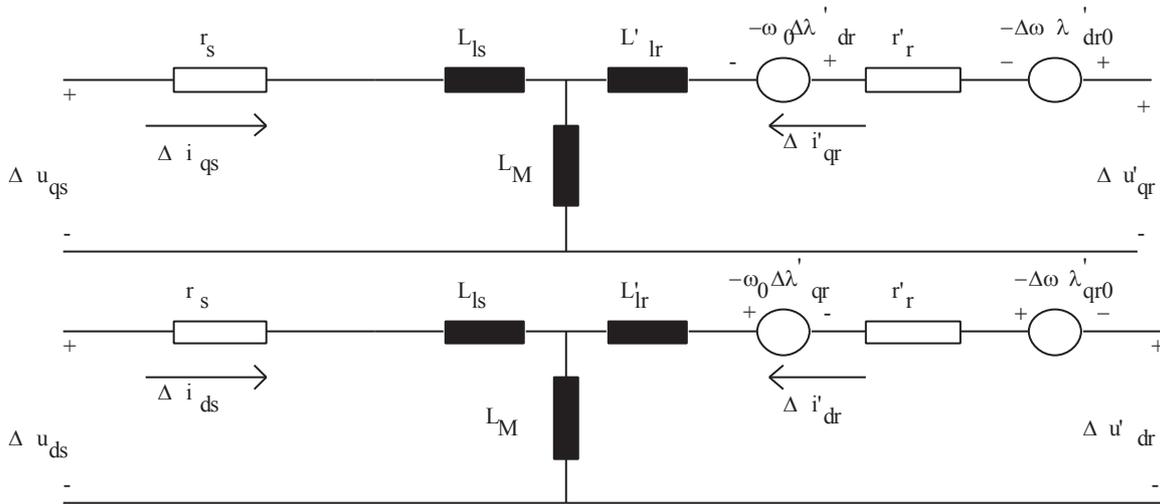

Fig. 6.1. Small-signal stationary reference-frame equivalent circuits for a three-phase, symmetrical induction machine

6.3. Single-phase induction machine linearisation

In the concrete case of the unsymmetrical single-phase induction machine, the flux linkage per second version is selected for compactness, and the input vector u is chosen to be the voltages vector. Comparing with the symmetrical three-phase induction machine case, the same general considerations are valid. The asymmetrical configuration of this induction machine type can be included readily in the state-variable equations form by using the turns' ratio value k.

The small-signal expression for electromagnetic torque can be deduced from the large-signal torque expression in two forms, depending on the selected set of state-variables:

I) Flux linkages models or currents models:

$$T_e = K \cdot \big(x_{1q} x_{2d} - x_{1d} x_{2q} \big) \tag{804}$$

II) Mixed flux linkages and currents models ($\underline{x}_1$ current space vector, $\underline{x}_2$ flux linkage space vector):

$$T_e = K \cdot \left(\frac{1}{k} x_{1q} x_{2d} - k x_{1d} x_{2q} \right) \tag{805}$$

where K denotes a constant corresponding to the selected set of state-variables. For example:

$$K = \frac{P}{2} \cdot \frac{x_m}{k\omega_b D} \quad \text{if the stator and rotor flux linkages are selected as state-variables.}$$

If we apply the small-displacement approach to the state-variables, it results:

I) Flux linkages models or currents models:

$$\Delta T_e = K \cdot \left(x_{1q0} \Delta x_{2d} + x_{2d0} \Delta x_{1q} - x_{1d0} \Delta x_{2q} - x_{2q0} \Delta x_{1d} \right) \tag{806}$$

II) Mixed flux linkages and currents models ($\underline{x}_1$ current space vector, $\underline{x}_2$ flux linkage space vector):

$$\Delta T_e = K \cdot \left(\frac{1}{k} x_{1q0} \Delta x_{2d} + \frac{1}{k} x_{2d0} \Delta x_{1q} - k x_{1d0} \Delta x_{2q} - k x_{2q0} \Delta x_{1d} \right) \tag{807}$$

For a complete description of the linearised induction machine model, we have to include also the mechanical equation, expressed in small-signal form:

$$p \frac{\Delta \omega_r}{\omega_b} = \frac{P}{2J} \cdot \left(\Delta T_e - \Delta T_L \right) \tag{808}$$

Following are the final elements , which describe the linearised model of the single-phase induction machine:

$$p x(t) = A \cdot x(t) + B \cdot u(t) + E \cdot z(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t) \tag{809-810}$$

where:

$$x(t) = \begin{bmatrix} \Delta x_{1q} & \Delta x_{1d} & \Delta x_{2q} & \Delta x_{2d} & \Delta \omega_r \end{bmatrix}^T \tag{811}$$

$$u(t) = \begin{bmatrix} \Delta u_{qs} & \Delta u_{ds} & \Delta u_{qr} & \Delta u_{dr} & \Delta T_L \end{bmatrix}^T \tag{812}$$

$$A = -M^{-1} \cdot S$$

$$B = M^{-1}$$

$$M = \begin{bmatrix} L & \mathbf{0}(4,1) \\ l_{21} & -\frac{2J}{P} \end{bmatrix}; \quad S = \begin{bmatrix} R & r_{12} \\ \mathbf{0}(1,4) & 0 \end{bmatrix} \tag{813-814}$$

$$l_{21} = K \cdot \begin{bmatrix} x_{2d0} & -x_{2q0} & -x_{1d0} & x_{1q0} \end{bmatrix} \text{ or } l_{21} = K \cdot \begin{bmatrix} \frac{1}{k} x_{2d0} & -k x_{2q0} & -k x_{1d0} & \frac{1}{k} x_{1q0} \end{bmatrix}$$

$$r_{12} = \begin{bmatrix} 0 & 0 & \frac{1}{k} \lambda_{dr0}(x_{1d0}, x_{2d0}) & -k \lambda_{qr0}(x_{1q0}, x_{2q0}) \end{bmatrix}^T ;$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5 \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{0}_5 \tag{815-816}$$

The perturbations vector can be extracted from the inputs vector as:

$$z(t) = \Delta T_L \text{ and } E = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{P}{2J} \end{bmatrix}^T \tag{817-818}$$

The eigenvalues can be computed either from the characteristic equation or by using the motional impedance matrix of the machine:

$$\det \left[Z_m(\lambda) \right] = \det \left[M\lambda + S \right] = 0 \tag{819}$$

In Fig. 6.2 is illustrated the small-signal equivalent circuit of the single-phase induction machine in stationary reference frame:
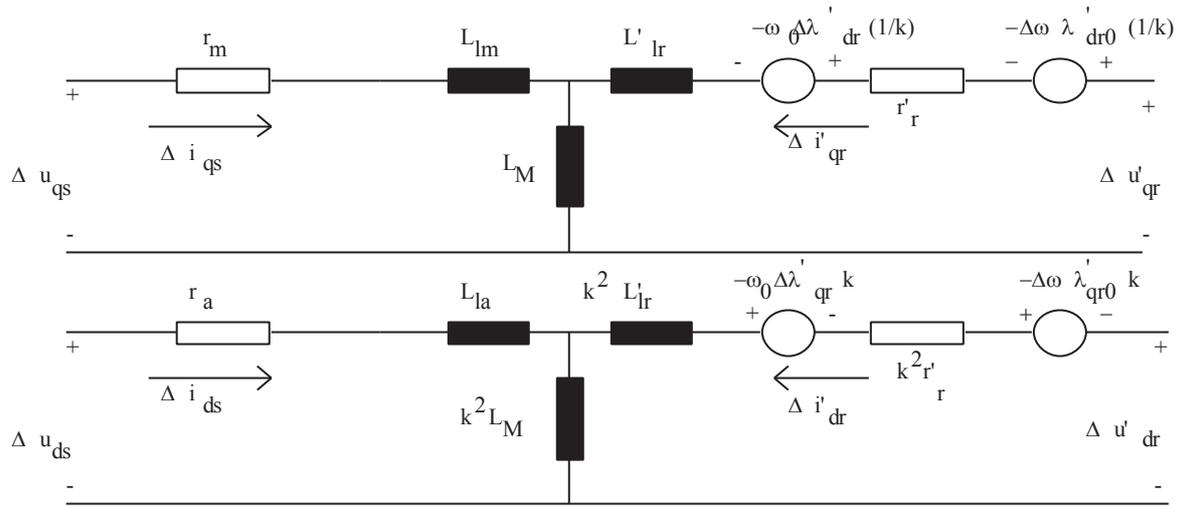
Fig. 6.2. Small-signal stationary reference-frame equivalent circuits for a
single-phase, unsymmetrical induction machine

References

[1] Adkins, B. 1957. "The General Theory of Electrical Machines", Chapman & Hall Ltd, London

[2] Alger, P.L. 1970. "Induction Machines. Their Behaviour and Uses", Second Edition, Gordon and Breach Science Publishing, New York

[3] Blaschke, F. 1971. "Das Prinzip der Feldorientierung die Grundlage fur die Transvektor - Regelung von Diehfeldmaschinen", Siemens-Zeitschrift 45, heft 10, pp. 757-760

[4] Blaschke, F. 1973. "Das Verfahren der Feldorientierung zur Reegelung der Drehfeldmaschine", Ph.D. dissertation, University of Braunschweig, Braunschweig, Germany

[5] Blaschke, F. 1972. "The Principle of Field Orientation as Applied to the New Transvector Closed Loop Control for Rotating Machines", Siemens Rev., Vol. 39, No. 5, pp. 217-220

[6] Bose, B.K. 1986. "Power Electronics and Drives", Prentice-Hall, Englewood Cliffs, New Jersey

[7] Chin, T-H. 1994. "Approaches for Vector Control of Induction Motor Without Speed Sensor", in Conf. Rec. IECON'94, Vol. 3, pp. 1616-1620

[8] Clarke, E. 1943. "Circuit Analysis of Power Systems - Vol. I, Symmetrical and Related Components", John Wiley & Sons, New York

[9] Correa, M.B.R., et al. 1998. "Field Oriented Control of a Single-Phase Induction Motor Drive", in Conf. Rec. Power Electronics Specialists, PESC'98, Fukuoka, Japan, Vol. II, pp. 990-996

[10] Correa, M.B.R., et al. 1999. "Single-phase Induction Motor Drives Systems", in Conf. Rec. Annual Power Electronics Conf., APEC'99, Anaheim, USA, pp. 403-409

[11] DeDoncker, R., Novotny, D.W. 1988. "The Universal Field Oriented Controller", in Conf. Rec. IEEE-IAS Annual Meeting, pp. 450-456

[12] DeDoncker, R., et al. 1995. "Comparison of Universal Field Oriented (UFO) Controllers in Different Reference Frames", in IEEE Trans. Power Electronics, Vol. 10, No. 2. pp. 205-213

[13] Diana, G., Harley, R.G. 1989. "An Aid for Teaching Field Oriented Control Applied to Induction Machines", in IEEE Trans. Power Systems, Vol. 4, No. 3, pp. 1258-1262

[14] Erdman, W.L., Hoft, R.G. 1990. "Induction Machine Field Orientation along Airgap and Stator Flux", in IEEE Trans. Energy Conv., Vol. 5, No. 1, pp. 115-121

[15] Fitzgerald, A.E., et al. 1990. "Electric Machinery" Fifth Ed., McGraw-Hill Publishing, New York

[16] Fortescue, C.L. 1918. "Method of Symmetrical Co-ordinates Applied to the Solution of Polyphase Network", AIEE Transactions, Vol. 37, pp. 1027-1115

[17] Franklin, G.F., et al. 1997. "Digital Control of Dynamic Systems", 3rd Edition, Addison-Wesley, New York

[18] Hasse, K. 1969. "Zur Dynamic Drehzahlgeregelter Antriebe Mit Stromrichter gespeisten Asynchron-Kuzschlublaufermaschinen", Ph. D. dissertation, Technische Hochschule Darmstadt, Darmstadt, Germany

[19] Holtz, J. 1995. "The Representation of AC Machine Dynamics by Complex Signal Flow Graphs", in IEEE Trans. Ind. Appl., Vol. 42, No. 3, pp. 263-271

[20] Holtz, J. 1998. "Sensorless Position Control of Induction Motors - an Emerging Technology", in Conf. Rec. AMC'98, Coimbra, Brazil, pp. 1-14

[21] Joetten, R., Maeder, G. 1982. "Control Methods for Good Dynamic Performance Induction Motor Drives Based on Current and Voltage as Measured Quantities", in Conf. Rec. IEEE Int. Semiconductor Power Converter Conf., Orlando, pp. 397-407

[22] Kelemen, A., Imecs, M. 1987. "Vector Control of AC Drives", G.J. Ritter, Budapest

[23] Kovacs, P.K., 1984. "Transient Phenomena in Electrical Machines", Elsevier Science Publihsers, Amsterdam

[24] Kovacs, P.K., Racz, L. 1959. "Transiente Vorgänge in Wechselstrommaschinen", Ungarischen Akademie der Wissenschaften, Budapest

[25] Krause, P.C., et al 1995. "Analysis of Electric Machinery", IEEE Press, New York

[26] Krause, P.C. 1965. "Simulation of Unsymmetrical 2-phase Induction Machines", in IEEE Trans. Power Apparatus and Systems, Vol. 84, No. 11, pp.1025-1037

[27] Krause, P.C., Thomas, C.H. 1965. "Simulation of Symmetrical Induction Machinery", in IEEE Trans. Power Apparatus and Systems, Vol. 84, No. 11, pp. 1038-1053

[28] Kuznetov, S.B. 1981. "Brushless Asynchronous Induction Machines", Ph.D. dissertation, University of London, London, UK

[29] Lai, Y-S. 1999. "Modelling and Vector Control of Induction Machines- a New Unified Approach", in Conf. Rec. Power Engineering Soc. Winter Meeting, Vol. I, pp. 47-52

[30] Lemaire-Semail, B., et al. 1991. "Modelling of Vector Controlled Cage Induction Motor with FEM", in IEE Proceedings B, Vol. 138, No. 6, pp. 297-302

[31] Leonhard, W., 1985. "Control of Electric Drives", Springer Verlag, New York

[32] Levi. E. 1984. "Polyphase Motors", Wiley, New York

[33] Lorenz, R.D., et al. 1994. "Motion Control with Induction Motors", in Proceedings of the IEEE, Vol. 82, No. 8, pp. 1215-1238

[34] Murata, T. et al. 1990. "Vector Control for Induction Machine on the Application of Optimal Control Theory", in IEEE Trans. Ind. Electronics, Vol. 37, No. 4, pp. 282-290

[35] Nabae, A. et al 1980. " An Approach to Flux Control of Induction Motors Operated with Variable-Frequency Power Supply", IEEE Trans. Ind. Appl. Vol.IA-16, No. 3, pp. 342-349

[36] Novotny, D.W., Lipo, T.A. 1996. "Vector Control and Dynamics of AC Drives", Oxford University Press, New York

[37] Ong, C-M. 1998. "Dymanic Simulation of Electric Machinery - Using Matlab/Simulink", Prentice Hall, Upper Saddle River, New Jersey

[38] Ohtani, T., et al. 1992. "Vector Control of Induction Motor without Shaft Encoder", in IEEE Trans. Ind. Appl., Vol. 28, No. 1, pp. 157-164

[39] Park, R.H. 1929. "Two-Reaction Theory of Synchronous Machines Generalised Method of Analysis - Part I", AIEE Transactions, Vol. 48, pp. 716-727

[40] Popescu, M., Navrapescu, V. 2000. "Modelling in Stationary Frame of Single and Two-phase Induction Machines Including the Effect of Iron Loss and Magnetising Flux Saturation" - in Proceedings of International Conference of Electrical Machines, ICEM 2000, 28-30 August, Espoo, Finland, Vol. I, pp. 407-411

[41] Popescu, M. 2000. "Analysis and Modelling of Single-phase Induction Motor with External Rotor for Domestic Applications" - in Proceedings of IEEE-IAS Annual Meeting, 8-12 October, Rome, Italy, Vol. I, pp. 463-470

[42] Rajashekara, K., et al. 1996. "Sensorless Control of AC Motor Drives" IEEE Press, New York

[43] Rogers, G.J. 1965. "Linearized Analysis of Induction Motor Transients", Proc. IEE, Vol. 112, pp. 1917-1926

[44] Rosenbrock, H.H. 1963. "Some General Implicit Processes for the Numerical Solution of Differential Equations", Computer Journal 5, pp. 329-330

[45] Salvatore, L., et al. 1998. "Vector Control of Induction Motors by Using EKF and CNF", in Conf. Rec. PEVD'98, Power Electronics and Variable Speed Drives, Seventh International Conference, London, UK, pp. 504-509

[46] Schauder, C. 1992. "Adaptive Speed Identification for Vector Control of Induction Motors without Rotational Transducers", in IEEE Trans. Ind. Appl. Vol. 28, No. 5, pp. 1054-1061

[47] Slemon, G.R. 1992. "Electric Machines and Drives", Addison Wesley, Reading

[48] Slemon, G.R. 1994. "Electrical Machines for Variable-Frequency", in Proceedings of the IEEE, Vol. 82, No. 8, pp.1123-1138

[49] Slemon, G.R. 1989. "Modelling of Induction Machines for Electric Drives", in IEEE Trans. Ind. Appl., Vol. 25, No. 6, pp. 1126-1131

[50] Stanley, H.C. 1938. "An Analysis of the Induction Motors", AIEE Transactions, Vol. 57 (Supplement), pp. 751-755

[51] Thiringer, T. 1996. "Measurements and Modelling of Low-Frequence Disturbances in Induction Machines" Ph. D. dissertation, Chalmers University of Technology, Göteborg, Sweden

[52] Trzynaldowski, A.M. 1994. "The field Orientation Principle in Control of Induction Motors", Kluwer Academic Publishers, Amsterdam

[53] Vainio, O., et al. 1992. "A Digital Signal Processing Approach to Real-Time AC Motor Modeling", in IEEE Trans. Ind. Appl., Vol. 39, No. 1, pp. 36- 45

[54] Vas, P. 1990. "Vector Control of AC Machines", Oxford University Press, Oxford

[55] Vas, P. 1992. "Electrical Machines and Drives", Oxford University Press, Oxford

[56] Vas, P: 1998. "Sensorless Vector and Direct Torque Control", Clarendon Press, Oxford

[57] Veinott, C.G. 1959. "Theory and Design of Small Induction Machines", McGraw Hill, New York

[58] Xu, X., et al. 1988. "A Stator Flux Oriented Induction Machine Drive", in Conf. Rec. Power Electronics Specialists, PESC'88, Kyoto, Japan, pp. 870-876.

[59] Xu, X., Nowotny, D.W. 1990. "Implementation of Direct Stator Flux Oriented Control on a Versatile DSP Based System", in Conf. Rec. IEEE-IAS Annual Meeting, pp. 437-443

[60] Xu, X. Nowotny, D.W. 1992. "Selection of the Flux Reference for Induction Machines in Field Weakening Regin", in IEEE Trans. Ind. Appl., Vol. 28, No.6, pp. 1353-1358

[61] Zhao, Y., Lipo, T.A. 1996. "Modeling and Control of a Multi-phase Induction Machine with Structural Unbalance" Part I, in IEEE Trans. Energy Conv., Vol. 11, No. 3, pp. 570-577

[62] Zhao, Y., Lipo, T.A. 1996. "Modeling and Control of a Multi-phase Induction Machine with Structural Unbalance" Part II, in IEEE Trans. Energy Conv., Vol. 11, No. 3, pp. 578-584

[63] *** 1999. "Vector Control of AC Induction Motors Using the ADMC 331" Whitepaper, Analog Devices Inc., USA